Graphical Associations and Codes with Small Covering Radius

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Contents

Declaration

All work in this thesis is my own except where otherwise stated. Theorem 2.12 and Lemma 3.2 are too simple to be considered original, although I could not find either explicitly stated anywhere.

Thanks

Thanks to my supervisors, John and Elizabeth for guiding me through this process. Especially to John who first introduced me to Coding Theory, put up with me during honours and then offered to take me on again.

To Brendan for offering me new insights into graph theory.

To my fellow students, its been great knowing you all.

To my family who don't really understand what I do, thanks for your support over the years.

To Pete, I owe you everything.

Abstract

Determining the minimum size of a set of objects required to cover a space is one of the classical problems of combinatorics. We look at codes with a minimum distance of 2 and and covering radius of 1. The focus is on codes over a symbol set of size 3 or 4. Graphs and codes are associated using 2 methods. Using the association methods, new techniques are developed to work towards the covering problem. Some new results are obtained, as well as new proofs of existing results.

Chapter 1

Introduction

1.1 The Problem

A combinatorial problem in Coding Theory is to find the minimum or maximum cardinality of a code with a given set of properties. The problem of finding $L_q(n, d)$, the maximum size of a code of length n over q symbols with a minimum distance at least d, is well studied. Results such as the Singleton and Hamming bounds give very good general limits. The problem of finding $K_q(n, R)$, the minimum cardinality of a code of length n over q symbols with a covering radius at most R, has also been studied, though not to the same extent. The sphere covering bound and Delsarte bound give good limits. The Norse bounds are good for binary codes and other results are good for self-dual codes.

A newer problem is to determine $K_q(n, R, d)$, the maximum cardinality of a code of length n over q symbols with minimum distance at least d and covering radius at most R.

It is obvious that

(1.1) $K_q(n, R, d) \geq K_q(n, R)$.

However it is an open problem to determine for what values of q, n, d and

R equation 1.1 has strict inequality.

Quistorff [Qui01] showed that for $n \leq 3$, $K_q(n,1,2) = K_q(n,1)$. Thus the next problem in lexicographic order is $K_q(4,1,2)$.

By the sphere covering bound $K_2(4,1) \geq 4$. $\mathcal{C} = \{0000, 0011, 1101, 1110\}$ is a $[4,4,2]_2 1$ covering code. Hence

(1.2) $K_2(4,1) = K_2(4,1,2) = 4$

By the sphere covering bound $K_3(4,1) \geq 9$. This can be achieved with the $[4,3,2]_4 1$ Hamming code. It has minimum distance 2 and so

(1.3) $K_3(4,1) = K_3(4,1,2) = 9$

With $q = 4$ things get more interesting. Stanton, Horton and Kalbfiesch [SHK69] showed that $K_4(4,1) = 24$, not 20 as is the minimum as given by the sphere covering bound. Ostergard, Quistorff and Wassermann [OQW05] showed via exhaustive computation that

(1.4) $K_4(4,1,2) = 28$.

and hence

(1.5) $K_4(4,1) < K_4(4,1,2)$

1.2 Scope of the Research

The aim of the research is to find a proof of equation 1.4 that does not require an exhaustive search; and in doing so develop techniques which may be used to solve similar problems. The aim of finding a new proof of inequality 1.4 is not achieved. However a new proof of inequality 1.5 is found and new techniques are developed.

Chapter 1 introduces all the basic coding theory and graph theory results required. These results and definitions can mostly be found in any standard text on the subject, e.g. [Hil86] [HP03].

Chapter 2 develops the techniques of association methods which are used throughout this research. Of particular significance are Theorem 2.6, which gives necessary and sufficient conditions for a simple graph to be associated with a code, and Theorem 2.14 which gives the conditions for general graphs. Various properties of graphs can then be used to find a variety of properties of codes. This also provides a useful tool for determining equivalence of codes.

Chapter 3 uses the the association methods of chapter 2 and develops the theory using simple graphs. Simple graphs make the machinery easier to develop, but restricts the results to $[n, V, n-1]_q$ codes. Theorem 3.3 [OQW05] shows that the size of the ball of radius 1 around a code can be determined by counting the number of multi-coloured triangles in the graph associated with the code. This is then used to develop bounds on the maximum size of the ball around the code, and hence the minimum size of a covering code.

Chapter 4 generalises the techniques of chapter 3. Using graphs with multiple edges between vertices allows for all codes to be represented using the association scheme. Most of the results of chapter 3 can be generalised to multiple edged graphs.

Chapter 5 applies the techniques and results of chapter 3 and 4 to investigate $K_q(4,1,2)$. This has previously been solved using an exhaustive computer search. A new proof is given that $K_q(4,1) < K_q(4,1,2)$.

Chapter 6 provides a brief summary of results and techniques of the research and outlines some further directions.

1.3 Preliminary Definitions

1.3.1 Coding Theory

Many of these results and definitions appear in any standard work on Coding Theory (eg.[Hil86] [HP03]).

Definition 1.1. *$d(u, v)$, the* Hamming distance *between any two $n-$tuples u and v, is the number of positions in which they differ.*

The Hamming distance forms a metric. It is the most important, and in most cases only metric used in coding theory.

Definition 1.2. Hamming space *$H(n, q)$ is a metric space consisting of all of the possible ordered n-tuples with symbols $0, 1, \ldots, q-1$, with the Hamming metric.*

Definition 1.3. *A code $\mathcal{C}$ is a subset of a Hamming space.*

The n-tuples in a Hamming space are called words. Those n-tuples which belong to the code in question are called codewords. Codewords will be represented as row vectors. $\mathcal{C}$ will always denote a code throughout this thesis.

Definition 1.4. *The* minimum distance *d of a code $\mathcal{C}$ is the minimum $d(u, v)$ such that $u, v \in \mathcal{C}$.*

Definition 1.5. *$B_r(v)$, the ball or radius r is the set of words at a Hamming distance at most r from the codeword v. $B_r(\mathcal{C})$ is the union of the balls of radius r around each code word so that,*

(1.6) $B_r(\mathcal{C}) = \{\cup_{v \in \mathcal{C}} B_r(v)\}$.

Definition 1.6. *The* covering radius *of an $[n, k, d]_q$ code $\mathcal{C}$ is the minimum R such that $B_R(\mathcal{C}) = H(n, q)$.*

Definition 1.7. *The* packing radius *of a code is the maximum r such that non-intersecting balls of radius r can be established around every codeword.*

The packing radius is linked to the minimum distance.

(1.7) $r = \left\lfloor \frac{d-1}{2} \right\rfloor$

Usually the packing radius is considered for its link to the error correcting capability of the code. However in this document we are primarily concerned with codes as combinatorial objects, and so make no further mention of error correcting.

The five most interesting properties of a code are summarised in the following standard notation.

Definition 1.8. *If $\mathcal{C}$ is an $[n, V, d]_q R$ code then, $\mathcal{C}$ is a code of length n over q symbols with V codewords, a minimum distance of d and covering radius of R.*

We may leave out d or R if talking about a more general class of codes. It is more common to use M(for modulus of code) instead of V, but we use V (for vertex) due to the connections we draw with graphs.

For simplicity we will always use the integers mod q as the symbol set. But we generally do not make use of the group properties of this set.

Definition 1.9. *Codes $\mathcal{C}$ and $\mathcal{D}$ are equivalent if they both have the same parameters, that is they are both $[n, V, d]_q R$ codes, and $\mathcal{D}$ can be obtained from $\mathcal{C}$ using any combination of the following operations.*

1. *Permute the symbols appearing in a fixed position of the code.*

2. *Permute the positions of the code.*

If two codes are equivalent, then they have the same structure, and are only superficially different. Therefore we only consider codes to be different if they are non-equivalent.

Definition 1.10. *A codeword u, r-covers a word v if $v \in B_r(u)$.*

In this document we are only concerned with balls of radius 1, so we will refer to a word being covered by u , if it is 1-covered by u.

Definition 1.11. *H_{is} is the hyperplane of the Hamming space H of all words which have symbol s in position i*

$$\mathcal{C}_{is} = \mathcal{C} \cap H_{is}$$

The following becomes a useful tool.

Definition 1.12. *Let $\mathcal{C}$ be an $[n, k, d]_q$ code. Then*

$$K_{is} = |H_{is} \setminus B_1(\mathcal{C})| + |\mathcal{C}_{is}|$$

.

Definition 1.13. *$p_q(n, V, d, r)$ is the maximum size of $B_r(\mathcal{C})$ for $\mathcal{C}$ an $[n, V, d]_q$ code.*

If $\mathcal{C}$ is an $[n, V, d]_q$ code then $|B_r(\mathcal{C})| \leq p_q(n, V, d, r)$.

The following are standard results and are included without proof as they can be found in any standard text on coding theory.

Theorem 1.14 (Singleton Bound). *([HP03] Thm 2.4.1) Let $\mathcal{C}$ be an $[n, V, d]_q$ code, then*

(1.8) $V \leq q^{n-d+1}$.

Theorem 1.15 (Hamming Bound). *([HP03] Thm 1.12.1, also called the sphere packing bound.) Let $\mathcal{C}$ be an $[n, V, d]_q$ code. Let $r = \lfloor \frac{d-1}{2} \rfloor$ then*

(1.9) $q^n \geq V \sum_{i=0}^{r} \binom{n}{i} (q-1)^r$.

Theorem 1.16 (Sphere Covering Bound). *([HP03] Thm 11.1.4) Let $\mathcal{C}$ be an $[n, V]_q R$ code. Then*

$$(1.10) \quad q^n \le V \sum_{i=1}^{R} \binom{n}{i} (q-1)^i.$$

Definition 1.17. *$K_q(n, r)$ is the minimum V such that an $[n, V]_q R$ code exists.*

Definition 1.18. *$K_q(n, r, d)$ is the minimum V such that an $[n, V, d]_q R$ code exists.*

Definition 1.19. *Let $\mathcal{C}$ be an $[n, V, d]_q$ code. Let $\widehat{\mathcal{C}}$ be the $[n-1, \widehat{V}, \widehat{d}]_q$ code obtained by deleting the i^{th} coordinate of every code word. We say that $\widehat{\mathcal{C}}$ has been obtained by* puncturing *$\mathcal{C}$ in the i^{th} position.*

1.3.2 Graph Theory

These definitions can all be found in any standard work on graph theory. e.g. [Wil79] [Bol79].

A graph is a collection of vertices (points), and edges (ordered pairs of points). A graph can be graphically represented as dots (vertices) and lines (edges). There are many different graphical representations of a graph which may look different. But they are the same graph if the set of edges and vertices is the same.

$\mathcal{G}$ will always denote a graph throughout this document. If $\mathcal{G}$ is a graph with V vertices, then $|\mathcal{G}| = V$.

Definition 1.20. *A* simple graph *is a graph with no more than one edge between any two vertices, and with no loops.*

Definition 1.21. *K_n is the complete graph on n vertices. Any two vertices in the K_n are connected by an edge.*

Lemma 1.22. *K_n contains n vertices and $\binom{n}{2} = \frac{1}{2}(n-1)n$ edges.*

K_3 is also called a triangle.

Definition 1.23. *An* empty *graph has vertices but no edges.*

Definition 1.24. *The* (object)-degree *of a vertex is the number of (objects) incident with it.*

This is not a standard definition, but is intuitive enough. For example we use the (triangle)-degree, (colour)-degree, (double edges)-degree and others. If no object is given, then it is assumed to be (edge)-degree. This is in keeping with the standard definition of degree.

Definition 1.25. *The* degree $\delta(v)$ *of a vertex v is the number of edges incident with v.*

Definition 1.26. *A k-*regular *graph is a graph such that every vertex has degree k.*

Definition 1.27. *A* clique *is a sub-graph such that any two vertices in the sub-graph are connected by an edge. A clique is a sub-graph equivalent to a complete graph.*

Definition 1.28. *A* coclique *is a sub-graph such that no two vertices in the sub-graph are connected by an edge. A coclique is a sub-graph equivalent to an empty graph.*

The vertices in a coclique may be incident with some edges, but those edges do not connect to any of the members of the coclique.

The edges (or vertices) of a graph can be divided into disjoint sets. Each set is labled with a colour. In the graphical representaion the edges (or vertices) are drawn in the colour of the set they belong to. In this document we are primarily concerned with edge colouring.

Definition 1.29. *An n-coloured graph is a graph in which the edges are any one of n colours.*

Definition 1.30. *Two n-coloured graphs $\mathcal{G}$ and $\mathcal{H}$ are* equivalent *if the colours in $\mathcal{G}$ may be permuted to obtain $\mathcal{H}$.*

In this thesis we are mostly concerned with $n = 3$ and $n = 4$. When drawing graphs associated with a code the following conventions will be adhered to.

If the graph has 3 colours then,

Colour	position in code
purple	1
green	2
black	3.

If the graph has 4 colours then,

Colour	position in code
blue	1
purple	2
green	3
black	4.

Chapter 2

Associated Graphs

We can associate a graph to a code in several ways. Each vertex of the graph represents a code word or an entry in a codeword; then edges between vertices represent some relationship between the code words or entries. We use 2 different association methods.

2.1 OQW-Association Method

2.1.1 Definitions

The first association method we will be using (due to Ostergaard, Quistorff and Wassermann [OQW05]) and will henceforth be refered to as the OQW-Association method.

Definition 2.1. *A graph $\mathcal{G}$ and a code $\mathcal{C}$ are* OQW-associated *if each vertex of $\mathcal{G}$ represents a codeword of $\mathcal{C}$, and an edge of colour i connects two vertices of $\mathcal{G}$ if and only if the two codewords of $\mathcal{C}$ represented have the same symbol in the i^{th} position.*

In figure 2.1 The purple lines are colour 1, the green lines are colour 2 and the black lines are colour 3. Note that the relationship expressed by the edges is transitive. The 4-clique in colour 1 is a 4-coclique in colours 2 and

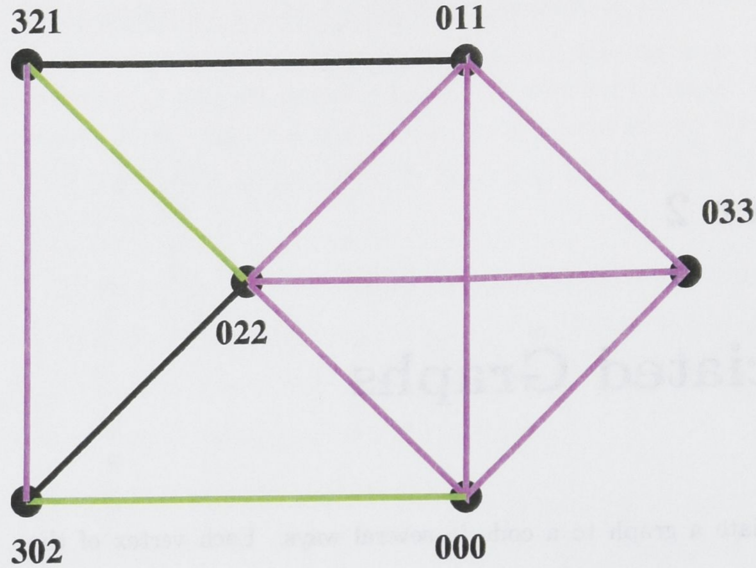

Figure 2.1: OQW-Association method (simple graph)

3. Each member of a coclique in colour 2 represents a different symbol in the 2nd position. Therefore the minimum number of symbols this code can be over is 4. In fact 4 is sufficient.

In figure 2.2 The blue lines represent colour 1, purple is colour 2, green is colour 3 and black is colour 4. Some of the code words are the same in 2 positions and are thus connected by 2 differently coloured edges (a double edge).

The graph of figure 2.1 can only be OQW-associated with one $[3, 6, 2]_4$ code up to equivalence. The graph in figure 2.2 can only be OWQ-associated with one $[4, 6, 2]_3$ code up to equivalence. This is true of all graphs constructed using the OQW association method as we will see in Theorem 2.6.

Lemma 2.2. *Let $\mathcal{C}$ be a code and $\mathcal{G}$ its OQW-associated graph. Then any code equivalent to $\mathcal{C}$ has a graph equivalent to $\mathcal{G}$ as its OQW-associated graph.*

Proof. Starting with a code $\mathcal{C}$, OQW-association constructs a graph $\mathcal{G}$, with

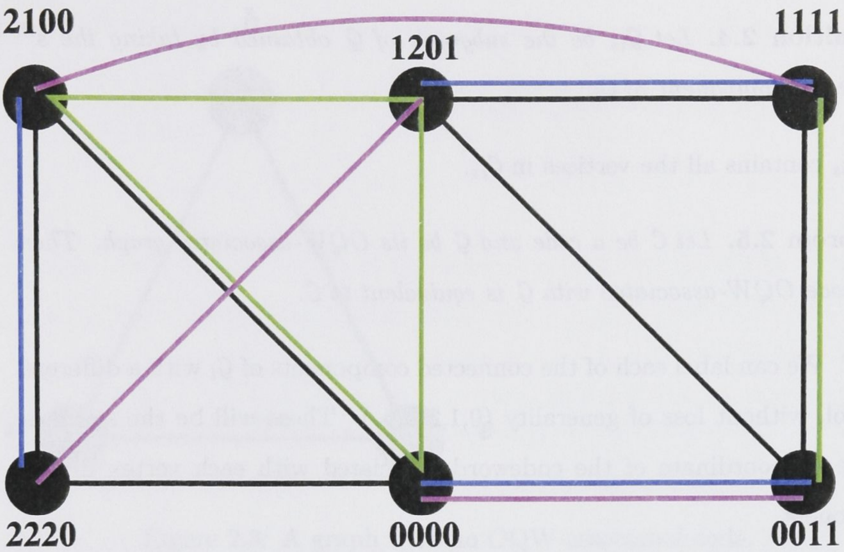

Figure 2.2: OQW-Association method (mulit-graph)

a set of vertices and edges uniquely determined by $\mathcal{C}$. Thus to each code a unique graph is OQW-associated.

Let $\mathcal{D}$ be a code equivalent to $\mathcal{C}$. To obtain $\mathcal{D}$ from $\mathcal{C}$ we can permute the symbols in a fixed position or permute the positions of the code. Permuting the symbols in a fixed position preserves the property of two codewords having the same symbol in a fixed position. Permuting the positions of the code is equivalent to permuting the colours of the graph which yields an equivalent graph.

♡

We will see later that we are only interested in the structure of colours within the graph. So we only consider graphs to be different if they are non-equivalent, just as we only consider codes to be different if they are non-equivalent.

Definition 2.3. *Let $\mathcal{G}_i$ be the sub-graph of $\mathcal{G}$ obtained by taking all the edges of colour i and all the vertices of $\mathcal{G}$.*

Definition 2.4. *Let $\mathcal{G}_{is}$ be the subgraph of $\mathcal{G}$ obtained by taking the s^{th} connected component of $\mathcal{G}_i$.*

$\mathcal{G}_{is}$ contains all the vertices in $\mathcal{C}_{is}$.

Theorem 2.5. *Let $\mathcal{C}$ be a code and $\mathcal{G}$ be its OQW-associated graph. Then any code OQW-associated with $\mathcal{G}$ is equivalent to $\mathcal{C}$.*

Proof. We can label each of the connected components of $\mathcal{G}_i$ with a different symbol, without loss of generality (0,1,2,3...). These will be the symbols in the i^{th} coordinate of the codeword associated with each vertex in the component.

Repeating this process for each colour forms a code. Let q be the largest number of disconnected components of any $\mathcal{G}_i$. Let n be the number of colours of edges. V the number of vertices. So any code associated with the graph of an $[n, V, d]_q$ code is also an $[n, V, d]_q$ code.

The operations permitted to obtain equivalent codes are

1. Permute the symbols appearing in a fixed position of the code.

2. Permute the positions of the code.

Taking our symbol set to be $\{0, 1, 2, \ldots q-1\}$, we can relabel any of the components of $\mathcal{G}_i$ with any permutation of the symbol set. This corresponds to operation 1.

We can arbitrarily choose which colour corresponds to which position in the code. This corresponds to operation 2.

This exhausts the possibilities of labelling $\mathcal{G}$ with codewords. Thus any code associated with $\mathcal{G}$ is equivalent to $\mathcal{C}$.

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Thus if a graph can be OQW-associated with a code, then it is OQW-associated with a unique equivalence class of codes.

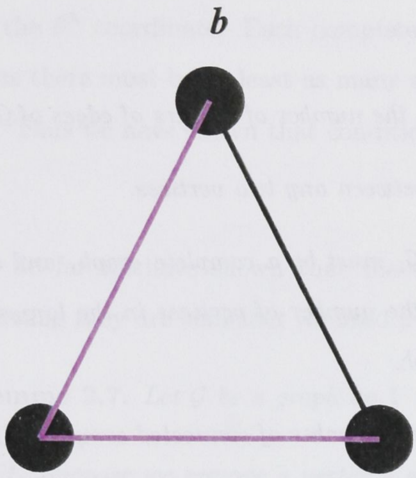

Figure 2.3: A graph with no OQW-associated code.

2.1.2 Properties

Not all graphs can have an OQW-associated code. For example figure 2.3 shows a graph with 2 colours of edges. Let purple be colour i, then vertices a and b have the same symbol in position i, vertices a and c have the same symbol in position i, which by transitivity would mean that vertices b and c should have the same symbol in position i; this is not so. Hence figure 2.3 cannot have an OQW-associated code.

There are many properties of a graph required for there to be a class of codes associated with it. Theorem 2.6 gives necessary and sufficient conditions that a graph must satisfy if it is to be associated with an $[n, V, n-1]_q$ code. We start with this small class of codes to make the graphs easier to manage. If we remove the restriction that $d = n - 1$ then there may be several edges between each vertex. Thus by restricting to $[n, V, n-1]_q$ we have restricted to simple graphs. In section 2.2 we will generalize this list of properties for all codes.

Theorem 2.6. *$\mathcal{G}$ is a graph OQW-associated with $\mathcal{C}$, an $[n, V, n-1]_q$ code, if and only if $\mathcal{G}$ has the following five properties.*

1. *$\mathcal{G}$ has V vertices.*

2. *n must be greater than or equal to the number of colours of edges of $\mathcal{G}$.*

3. *There is no more than one edge between any two vertices.*

4. *Any connected sub-graph of any $\mathcal{G}_i$ must be a complete graph, and q must be greater than or equal to the number of vertices in the largest single coloured connected sub-graph.*

5. *q must be greater than or equal to the number of connected components of any $\mathcal{G}_i$, $i \in \{0, 1, \ldots n\}$.*

First we show that these conditions are necessary. Further results are required to show sufficiency.

Proof. 1. The vertices of the graph and words of the code are in a bijective correspondence.

2. Each colour of edges represents a specific coordinate of the code. Therefore the length of the code can be no less than the number of colours of edges.

3. The minimum distance between any two words is $n - 1$. That is any two codewords must be different in at least $n - 1$ positions, and thus can be the same in at most 1 position. Thus there can only be a single edge between any two vertices.

4. The relationship expressed by edges is transitive, and so any vertices connected by a single coloured path, must be connected by an edge of that colour.

Given that any two codewords must differ in $n - 1$ places, there can be at most q codewords with the same symbol in position i. Hence a maximum of q vertices in a single coloured clique.

5. From 4 we know that $\mathcal{G}_i$ contains disjoint complete graphs. Each of these complete graphs represents the set of codewords with the same symbol

in the i^{th} coordinate. Each complete graph must have a different symbol, thus there must be at least as many symbols as disjoint complete graphs.

Thus we have shown that conditions 1-5 are necessary.

♡

So far we have shown that the five conditions are necessary. Before showing they are sufficient we need 2 further results.

Lemma 2.7. *Let $\mathcal{G}$ be a graph on V vertices that satisfies conditions 1-5.*

1. *Suppose we remove a vertex v and any edges including v to form $\mathcal{G}'$,*

2. *Suppose we add a vertex v and some edges (of any colour) such that any connected sub-graph of $\mathcal{G}'_i$ is complete and $\mathcal{G}'$ is a simple graph,*

then $\mathcal{G}'$ will also satisfy conditions 1-5.

Proof. 1. Conditions 1 and 3 are obvious. The number of colours in $\mathcal{G}'$ cannot have increased so condition 2 still holds. All connected sub-graphs of $\mathcal{G}'_i$ are complete; and cannot be larger than the connected sub-graphs in $\mathcal{G}_i$ thus condition 4 still holds. As we are removing a vertex, the number of connected components of $\mathcal{G}'_i$ cannot be larger than of $\mathcal{G}_i$ thus condition 5 still holds.

2. Conditions 1 and 3 are obvious. If edges of new colours are added then $n' \geq n$, to meet condition 2. By our construction conditions any connected sub-graph of $\mathcal{G}'_i$ must be complete; if a connected component of $\mathcal{G}'_i$ is larger than that of $\mathcal{G}_i$ then $q' \geq q$ to meet condition 4. If the addition of v causes the number of components in $\mathcal{G}'_i$ to be greater than in $\mathcal{G}_i$ for any i then $q' \geq q$ may be raised to meet condition 5.

In conclusion the values of n, V, and q may be different to n', V' and q', but conditions 1-5 are still met.

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Lemma 2.8. *Let $\mathcal{G}$ be a graph OQW-associated with an $[n, V, n-1]_q$ code, $\mathcal{C}$.*

1. *Suppose we remove a vertex v and any edges including v to form $\mathcal{G}'$,*

2. *Suppose we add a vertex v and some edges (of any colour) such that any connected sub-graph of $\mathcal{G}'_i$ is complete and $\mathcal{G}'$ is a simple graph,*

then $\mathcal{G}'$ will have an OQW-associated code $\mathcal{C}'$.

As we are dealing with codes in a strictly combinatorial sense, adding or removing a word will not effect the rest of the code as would be the case with a linear code.

Proof. 1. Let $\mathcal{G}'$ be formed by removing a vertex and the edges incident with it from $\mathcal{G}$. This removes a codeword from the OQW-associated code, $\mathcal{C}$ to form $\mathcal{C}'$ an $[n', V-1, n'-1]_{q'}$ code with $n' \leq n$ and $q' \leq q$.

2. Take $\mathcal{G}$, add a vertex v, and for each $i \in \{1, n\}$ add sufficient edges, including v, such that each connected component of $\mathcal{G}'_i$ is a complete graph. This adds a word to $\mathcal{C}$ to form $\mathcal{C}'$. The edges will determine what that word is. Provided every connected component of each $\mathcal{G}'_i$ is a complete subgraph, the word is uniquely determined by the edges of the associted vertex.

As the cliques of each colour may be larger, and there may be more colours than in $\mathcal{G}$, $q' \geq q$ and $n' \geq n$.

Then $\mathcal{C}'$ is an $[n', V+1, n'-1]_{q'}$ code.

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Now we can show that the 5 conditions are sufficient.

Proof of Thm 2.6. We proceed by induction on V.

Begin with $V = 1$; K_1 the graph consisting of one vertex. This satisfies all the required conditions and has an OQW-associated code. Suppose every graph $\mathcal{G}$ which satisfies conditions 1-5 with $V \leq m$ vertices has an OQW-associated code.

Take a graph $\mathcal{G}$ with $m+1$ vertices that satisfies conditions 1-5. Let $\mathcal{G}'$ be a graph obtained by removing a vertex from $\mathcal{G}$, and the edges including that vertex. From Lemma 2.7 we know that $\mathcal{G}'$ will also satisfy conditions 1-5. By the inductive hypothesis $\mathcal{G}'$ has an OQW-associated code.

Lemma 2.8 shows that when we add back the vertex and edges to form $\mathcal{G}$, then $\mathcal{G}$ must have an OQW-associated code.

Thus by induction every graph which satisfies conditions 1-5 has an OQW-associated graph.

$\heartsuit$

Thus the 5 properties of Theorem 2.6 are necessary and sufficient. And so results on the existence of graphs with the properties we require can be directly translated to results about codes. The following are some observations on OQW-associated graphs and.

From counting the number of colours and vertices of $\mathcal{G}$ we can determine the length and size of the OQW-associated code.

If there are more words in the code than symbols, then by the pigeon hole principle there must be at least two words with the same symbol in the i^{th} coordinate, and hence there must be at least one edge of colour i. This is true for each of the positions and hence there must be at least one edge of each colour. Thus the number of colours of edges must be exactly the length of the code; if $q < V$ then n equals the number of colours of edges. Also q is equal to the largest number of disconnected subgraphs in a $\mathcal{G}_i$ or the largest size of a single coloured connected sub-graph, which ever is the larger. If n is greater than the number of colours of edges in $\mathcal{G}$ then $q = V$.

Because we can permute symbols in any position of the code, the largest number of symbols required for a particular coordinate is the number of symbols required for the code. This is given by parts 4 and 5.

If two sides of a triangle are of colour i, then by transitivity, the third side must also be of colour i. Therefore there can be no 2-coloured triangles. Thus any triangle in an OQW-associated graph is either single coloured, or 3-coloured.

The following corollaries are mostly derived from parts 5 and 4 of Theorem 2.6. They give us a bit more information about the properties of the graphs we are dealing with.

Corollary 2.9. *If $\mathcal{G}$ is a graph OQW-associated with a $[n, V, n-1]_q$ code then $V \leq q^2$.*

Proof. The number of vertices in $\mathcal{G}_i$ is the same as the number of vertices in $\mathcal{G}$. The number of connected components of $\mathcal{G}_i$ must be less than or equal to q (Theorem 2.6 5). The number of vertices in each connected component of $\mathcal{G}_i$ must be less than or equal to q (Theorem 2.6 4). Thus the number of vertices in $\mathcal{G}_i$ must be less than q^2.

♡

This result can also be obtained by applying the Singleton Bound. (Theorem 1.14)

Corollary 2.10. *If $\mathcal{G}$ is a graph OQW-associated with an $[n, V, n-1]_q$ code, and let $V = zq + w$ where $z \in \mathbb{Z}^+$ and $0 \leq w < q$. Then there are at least*

$$\frac{q}{2}(z-1)z + wz \tag{2.1}$$

edges of each colour.

Proof. The minimum number of edges occurs when there are complete graphs of lowest degree. The connected components of $\mathcal{G}_i$ should all have as close to the same number of vertices as possible. *e.g.* Figure 2.4 shows the minimum number of edges of $\mathcal{G}_i$ for an $[n, 13, 2]_4$ code.

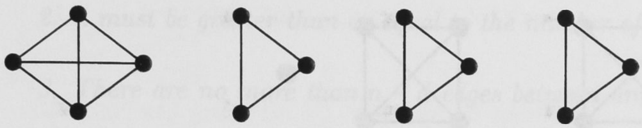

Figure 2.4: Minimum number of edges

There will be q disconnected components of $\mathcal{G}_i$ and if we have the minimum number of edges then each component will have at least z vertices, with w of the components having $z+1$ vertices.

Each complete graph on z vertices has

$$(2.2) \quad \frac{1}{2}(z-1)z$$

edges, and there are q of them. There are w of the components with an extra vertex, and hence an extra z edges.

♡

Corollary 2.11.

If $\mathcal{G}$ is a graph OQW-associated with an $[n, V, n-1]_q$ code and $V = zq + w$ then there are at most

$$(2.3) \quad \frac{zq(q-1)}{2} + \frac{w(w-1)}{2}$$

edges of each colour.

Proof. The largest number of edges for a given V will occur when we have complete graphs of the largest possible size, q. *e.g.* figure 2.5 shows the maximum number of edges of $\mathcal{G}_i$ for an $[n, 13, 2]_4$ code.

We can have z complete graphs on q vertices, and then a complete graph on w vertices.

♡

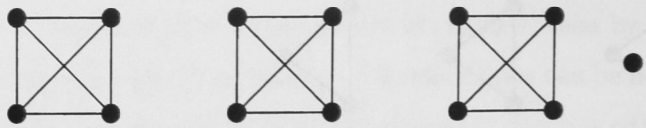

Figure 2.5: Maximum number of edges

2.1.3 Other useful ideas

We have been dealing with coloured graphs, but results which hold for graphs without colourings still hold when we introduce colours. The following are useful results which put bounds on the number of triangles (3-cliques) that a graph can have. They are included without proof, as the proofs do not demonstrate anything new.

Theorem 2.12. *Let $\mathcal{G}$ be a connected simple graph. If v is a vertex of degree δ then the maximum number of triangles that v can be incident with is $\binom{\delta}{2}$. This can be achieved if and only if v is part of a $(\delta + 1)$ clique.*

Theorem 2.13. *[NS63](Nordhaus, Stewart, J. McKay 1963) If $\mathcal{G}$ is a connected simple graph with V vertices E edges and T triangles then*

(2.4) $3TV \geq E(4E - V^2)$.

2.2 Generalizing the OQW-Association Method

We would like to extend Theorem 2.6 to be useful for a greater variety of codes. This does not require an extension of the definition of the OQW-association method. If $\mathcal{C}$ is an $[n, V, d]_q$ code then properties 1, 2, and 5 of theorem 2.6 remain the same. Properties 3 and 4 must be amended.

Theorem 2.14. *$\mathcal{G}$ is a graph OQW-associated with $\mathcal{C}$, an $[n, V, d]_q$ code if and only if $\mathcal{G}$ has the following five properties.*

1. *$\mathcal{G}$ has V vertices.*

2. *n must be greater than or equal to the number of colours of edges of $\mathcal{G}$.*

3. *There are no more than $n-d$ edges between any two vertices.*

4. *Any connected sub-graph of any $\mathcal{G}_i$ must be complete, and q^{n-d} must be greater than or equal to the number of vertices in the largest single coloured connected sub-graph.*

5. *q must be greater than or equal to the number of disconnected components of any $\mathcal{G}_i$, $i \in \{0, 1, \ldots, n\}$.*

Proof. 1,2 and 5, and sufficiency of these conditions see proof of Theorem 2.6.

3. Any two code words must differ in at least d positions, and hence can be the same in at most $n-d$ positions.

4. The subcode consisting of all the words with an a in the i^{th} position can be thought of as an $[n-1, V', d]_q$ code. By the Singleton bound this can have at most $q^{n-1-d+1} = q^{n-d}$ code words. Thus any single coloured connected component of $\mathcal{G}$ can have no more than q^{n-d} vertices.

♡

Now we no longer have simple graphs and our multi-graphs are allowed multiple edges (of different colours) between two vertices, but no loops.

Corollary 2.10 still holds, as each colour of edges must still adhere to property 5 of Theorem 2.6. Corollaries 2.9 and 2.11 must be amended.

Corollary 2.15. *If $\mathcal{G}$ is a graph OQW-associated with an $[n, V, d]_q$ code then $q^{n-d+1} \geq V$.*

Proof. The number of vertices in $\mathcal{G}_i$ is the same as the number of vertices in $\mathcal{G}$. The number of connected components of $\mathcal{G}_i$ must be less than or equal to q (Theorem 2.14 5). The number of vertices in each connected component of $\mathcal{G}_i$ must be less than q^{n-d} (Theorem 2.14 4). Thus the number of vertices in $\mathcal{G}_i$ must be less than q^{n-d+1}.

♡

As with Corollary 2.9, this can be obtained by applying the Singleton bound.

Corollary 2.16. *If $\mathcal{G}$ is a graph OQW-associated with an $[n, V, d]_q$ code, and $V = zq^{n-d} + w$, then there are at most*

$$\frac{zq^{n-d}(q^{n-d}-1)}{2} + \frac{w(w-1)}{2} \tag{2.5}$$

edges of each colour.

Proof. The largest number of edges for a given V will occur when we have complete graphs of the largest possible size that is q^{n-d} vertices (Theorem 2.14 part 5). We can have z q^{n-d}-cliques, and then a complete graph on w vertices.

♡

2.3 BEHL-Association Method

This association method is due to Blokhuis, Enger, Hollmann and van Lint [BEHvL01].

Definition 2.17. *If C is an $[n, V, d]_q$ code then a graph $\mathcal{G}$ is BEHL-associated in the following way: $\mathcal{G}$ contains n q-cocliques L_1 to L_n: each vertex represents a member of the symbol set, and each coclique represents a position in the code. There is an edge from $a \in L_i$ to $b \in L_j$ if and only if there is a codeword $v = (v_1, v_2, ..., v_n)$ such that $v_i = a$ and $v_j = b$.*

Figure 2.6 shows the BEHL associated graph for $C = \{(000), (011), (322)\}$.

Let $\mathcal{G}_c$ be the sub-graph representing the codeword c, then $\mathcal{G}_c$ is an n-clique.

The graph is simple, therefore an edge may be in more than one n-clique. Each word is represented by an n-clique, and so $\mathcal{G}$ has a maximum

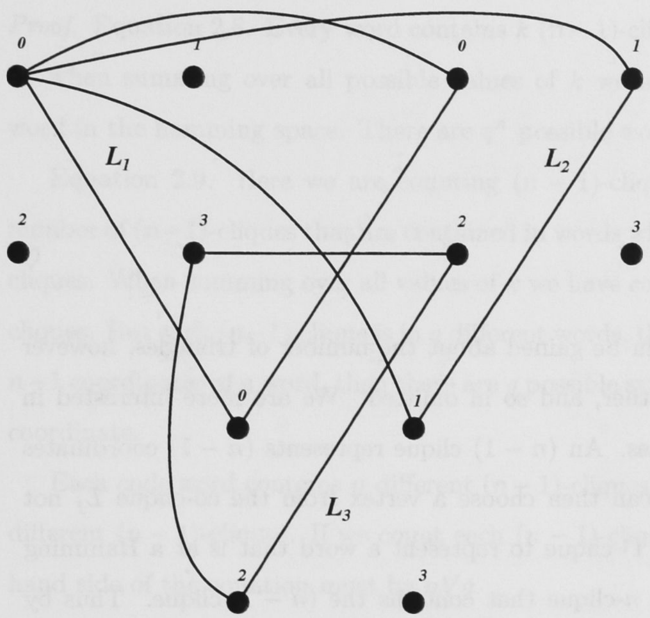

Figure 2.6: BEHL-association method

of $\frac{1}{2}n(n-1)V$ edges. This association method has the advantage that every word in the Hamming space can be represented by the sub-graph obtained by choosing an appropriate vertex from each co-clique and any edges connecting those vertices.

Lemma 2.18. *Let $\mathcal{G}$ be a graph BEHL-associated with an $[n, V, d]_q$ code. Then the minimum number of edges of $\mathcal{G}$ is*

$$\frac{Vn(n-1)}{2q^{n-d-2}} \tag{2.6}$$

Proof. Each word is a n-clique and so has $\frac{1}{2}n(n-1)$ edges. Any two code words can have at most $n-d$ symbols in common, thus any two n-cliques can have $n-d$ vertices in common. There can be at most q codewords that are the same in any $n-d$ positions. (Because they must all be different in the other d positions). Thus each $(n-d)$-clique can be in at most q n-cliques. An edge can be in at most q^{n-d-2} n-cliques. Thus we can divide the maximum number of edges by q^{n-d-2}. Thus there can be a minimum of

$$(2.7) \quad \frac{Vn(n-1)}{2q^{n-d-2}}$$

edges in $\mathcal{G}$.

♡

A similar result can be gained about the number of triangles, however it is not used any further, and so in omitted. We are more interested in counting $(n-1)$-cliques. An $(n-1)$ clique represents $(n-1)$ coordinates of a code word. We can then choose a vertex from the co-clique L_j not contained in the $(n-1)$-clique to represent a word that is at a Hamming distance of 1 from an n-clique that contains the $(n-1)$-clique. Thus by counting the number of $(n-1)$-cliques we can determine the size of the ball of radius 1 around a code.

We can develop some useful results counting these objects in different ways.

Definition 2.19. *Let S_k be the number of words in $H(n,q)$ (not necessarily code words) that when represented by an BEHL-associated graph $\mathcal{G}$ contain k $(n-1)$-cliques of $\mathcal{G}$.*

If $d \geq 2$ then no two codewords can contain the same $(n-1)$-clique. However two words may share an $(n-d)$-clique.

Lemma 2.20. *[BEHvL01] If $\mathcal{C}$ is an $[n,V,d]_q$ code where $d \geq 2$ then*

$$(2.8) \quad \sum_{k=1}^{n} S_k = q^n$$

and

$$(2.9) \quad \sum_{k=1}^{n} kS_k = Vnq.$$

Proof. Equation 2.8. Every word contains k $(n-1)$-cliques, for some $k \geq 0$. So when summing over all possible values of k we include every possible word in the hamming space. There are q^n possible words.

Equation 2.9. Here we are counting $(n-1)$-cliques. kS_k counts the number of $(n-1)$-cliques that are contained in words which contain k $(n-1)$-cliques. When summing over all values of k we have counted all the $(n-1)$-cliques. But each $(n-1)$ clique is in q different words, this is akin to choosing $n-1$ coordinates of a word, then there are q possible symbols to fill the other coordinate.

Each code word contains n different $(n-1)$-cliques, and so there are Vn different $(n-1)$-cliques. If we count each $(n-1)$-clique q times then right hand side of the equation must be nVq.

♡

Lemma 2.21. *[BEHvL01] If $\mathcal{C}$ is an $[n, V, d]_q$ code with $d \geq 2$ then*

$$\sum_{k=1}^{n} \binom{k}{2} S_k \geq \binom{n}{2} \frac{V^2}{q^{n-2}}. \tag{2.10}$$

Proof. We now count pairs of $(n-1)$-cliques within some word. If a word has k $(n-1)$-cliques, then it has $\binom{k}{2}$ pairs of $(n-1)$-cliques.

Let w be a word in the Hamming space $H(n, q-2)$. Let $a_w^{(i,j)}$ be the number of code words such that deleting the coordinates v_i and v_j yields w. Given that $d \geq 2$, $a_w^{(i,j)} \leq q$ for each combination of i, j, w. If we keep i, j fixed, then summing over all words in $H(n, q-2)$, each word will be counted exactly once. Hence

$$\sum_{w \in H(n,q-2)} a_w^{(i,j)} = V, \tag{2.11}$$

if we count the number of pairs of $(n-1)$ cliques in some word in $H(n, q)$. Removing the i^{th} coordinate of a word is the same as choosing an $(n-1)$-clique. Thus removing 2 coordinates is the same as choosing a pair of $(n-1)$-cliques within the same codeword. So the number of pairs of $(n-1)$-cliques within a word in $H_(n, q)$ is

$$\text{(2.12)} \quad \sum_{k=1}^{n} \binom{k}{2} S_k = \sum_{1 \geq i \geq j \geq n} \sum_{w \in H(n,q-2)} \binom{2a_w^{(i,j)}}{2}.$$

Using equations 2.11, 2.12, the Cauchy-Schwarz inequality and that $\binom{n}{2} \geq n^2$ for $n \geq 0$ we get that for each pair i, j

$$\text{(2.13)} \quad \sum_{k=1}^{n} \binom{k}{2} S_k \geq \binom{n}{2} \frac{V^2}{q^{n-2}}.$$

♡

These results will be used in subsequent chapters.

Chapter 3

$[3, V, 2]_q 1$ Codes

In this chapter we investigate the bounds on $|B_1(\mathcal{C})|$. If we keep n, d and q fixed and find an upper bound on $|B_1(\mathcal{C})|$, then we can find a lower bound on $K_q(n, R, d)$. We are using simple OQW-associated graphs and so restrict attention to $[n, V, n-1]q$ codes. The case $n \geq 4$ is dealt with easily. The remainder of the chapter develops graph theoretic techniques to deal with $[3, V, 2]_q$ codes.

3.1 $n \geq 4$

If $\mathcal{C}$ is an $[n, V, n-1]_q$ code for $n-1 \geq 3$ then the balls of radius 1 around the codewords do not intersect. Thus the size of a ball of radius 1 is $1 + n(q-1)$, and so the union of all the balls is:

(3.1) $|B_1(\mathcal{C})| = (1 + n(q-1))V.$

So for $\mathcal{C}$, an $[n, V, n-1]_q$ code with $n \geq 4$, we have found the maximum size of $B_1(\mathcal{C})$.

Theorem 3.1.

(3.2) $p_q(n, V, n-1, 1) = (1 + n(q-1))V \quad \forall n \geq 4$

As we have dispensed with the case of $[n, V, n-1]_q$ codes where $n > 3$ we can look at the more difficult problem of $[3, V, 2]_q$ codes. The case $q = 4$ has been most thoroughly investigated. Codes with minimum distance 2 are not very interesting from a coding theory point of view as they have only small error correcting ability. However finding covering sets is one of the classic problems of combinatorics [CHLL97].

3.2 Upper Bound

Lemma 3.2. *Let $\mathcal{C}$ be a $[3, V, 2]_q$ code, then*

(3.3) $|B_1(\mathcal{C})| \leq q^3$

Proof. The ball around the space can be no larger than the space in which the code is contained. ♡

Theorem 3.3. *[OQW05] Theorem 3. Let $\mathcal{C}$ be a $[3, V, 2]_q$ code with $\mathcal{G}$ its OQW-associated graph having V vertices and E edges. Let T_c be the number of multi-coloured triangles in $\mathcal{G}$, then*

(3.4) $|B_1(\mathcal{C})| = (1 + 3(q-1))V - 2E + T_c$

Proof. Using the principle of inclusion-exclusion: Let $u, v, w \in \mathcal{C}$

$$\begin{aligned} B_1(u) \cup B_1(v) \cup B_1(w) &= B_1(u) + B_1(v) + B_1(w) \\ &\quad - B_1(u) \cap B_1(v) - B_1(v) \cap B_1(w) - B_1(u) \cap B_1(w) \\ &\quad + B_1(u) \cap B_1(v) \cap B_1(w) \end{aligned}$$

$|B_1(v)| = 1 + n(q-1)$ for all words in $\mathcal{C}$. Thus the first "‘inclusion"’ is $(1 + n(q-1))V$.

If two words $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$ are joined by an edge, then $d(u, v) = 2$ and one of $u_1 = v_1$, $u_2 = v_2$ or $u_3 = v_3$. Without loss of generality we can assume $u_1 = v_1$. Then the words

(3.5) $\{(u_1, v_2, u_3), (u_1, u_2, v_3)\} = B_1(u) \cap B_1(v)$.

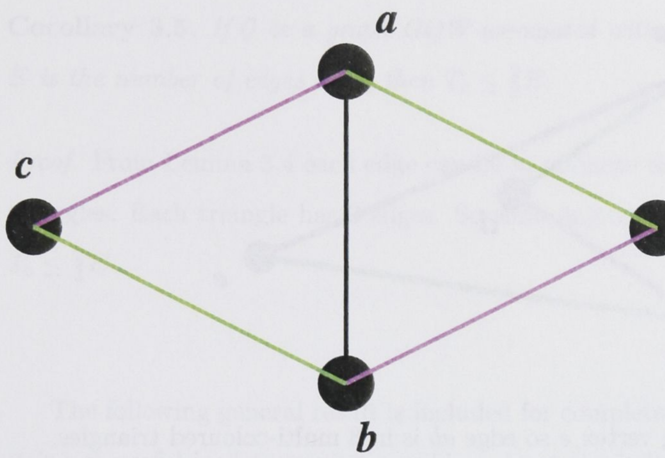

Figure 3.1: Edge ab is in 2 multi-coloured triangles

Every edge of $\mathcal{G}$ reduces $|B_1(\mathcal{C})|$ by 2. Thus the first "‘exclusion"' is $2E$.

If $\mathcal{G}$ has edges forming a multi-coloured triangle, then there are vertices $u = (u_1, u_2, u_3)$, $v = (u_1, v_2, v_3)$ and $w = (w_1, v_2, u_3)$. $\{(u_1, v_2, u_3)\} = B_1(u) \cap B_1(v) \cap B_1(w)$. Thus every multi-coloured triangle in $\mathcal{G}$ increases $|B_1(\mathcal{C})|$ by 1.

And so if u, v, w form a multi-coloured triangle then.

(3.6) $B_1(u) \cup B_1(v) \cup B_1(w) = 3(1 + n(q-1)) - 3 \times 2 + 1$

Thus our second "‘inclusion"' is T_c

If 2 vertices u, v are not connected by an edge then $B_1(u) \cap B_1(v) = \emptyset$. We only need to go as far as counting triangles because for any 4 codewords $u, v, w, x \in \mathcal{C}$ $B_1(u) \cap B_1(v) \cap B_1(w) \cap B_1(x) = \emptyset$.

Putting the inclusions and exclusions together we get equation 3.4.

♡

Lemma 3.4. *(with B.D.McKay) Let $\mathcal{G}$ be a graph OQW-associated with a $[3, V, 2]_q$ code. Then each edge can be in a maximum of 2 multi-coloured triangles.*

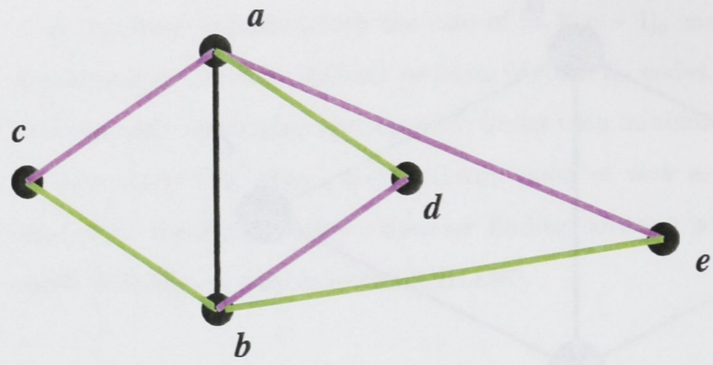

Figure 3.2: Adding vertex e so edge ab is in 3 multi-coloured triangles.

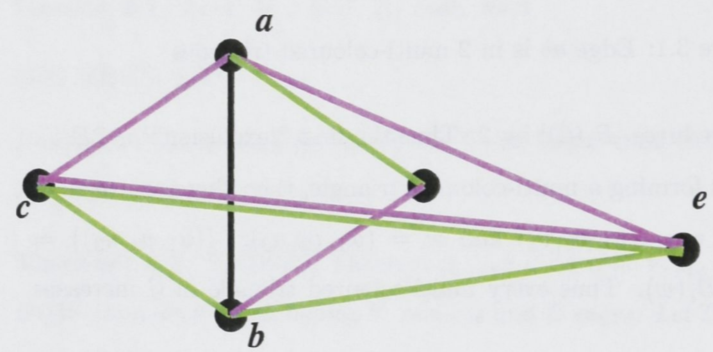

Figure 3.3: Due to transitivity a double edge is required.

Proof. Let an edge (ab) be in 2 multi-coloured triangles. Then there must be edges surrounding it of the structure shown in figure 3.1. From this we add another multi-coloured triangle to edge ab as shown in figure 3.2.

Since the edges represent a transitive relationship, a double edge is required to form single coloured triangles with vertex c (figure 3.3).

This contradicts part 3 of Theorem 2.6, and would mean that the minimum distance of the code is 1.

♡

Note this result only holds for $n = 3$, as the proof relies on having only 3 colours of edges.

Corollary 3.5. *If $\mathcal{G}$ is a graph OQW-associated with a $[3, V, 2]_q$ code and E is the number of edges in $\mathcal{G}$, then $T_c \leq \frac{2}{3}E$.*

Proof. From Lemma 3.4 each edge can be in no more than 2 multi-coloured triangles. Each triangle has 3 edges. So summing over all the edges we get $T_c \leq \frac{2}{3}E$.

♡

The following general result is included for completeness. Unfortunately it is not useful in determining anything about the ball around the code as explained by equation 3.1, but is interesting from a graph theory point of view.

Theorem 3.6. *Let $\mathcal{G}$ be a code OQW-associated with an $[n, V, n-1]_q$ code then each edge can be in at most $2\binom{n-1}{2}$ multi-coloured triangles.*

Proof. We have seen previously with $n = 3$ that the maximum number of multi-coloured triangles any edge can be in is 2. If we choose an edge to be of colour 1, then there are $n - 1$ colours to chose from. If we choose any two of them, then we can make at most 2 multi-coloured triangles. There are $\binom{n-1}{2}$ ways of doing this.

♡

We can combine these results together with results from chapter 2 to get an upper bound on $|B_1(\mathcal{C})|$.

Theorem 3.7. *Let $\mathcal{C}$ be a $[3, V, 2]_q$ code. Let $V = zq + w$ with $z \geq 0$ and $0 \leq w < q$ then*

$$|B_1(\mathcal{C})| \leq (1 + 3(q-1))V - \frac{4}{3}\left(\frac{3q}{2}(z-1)z + zw\right). \tag{3.7}$$

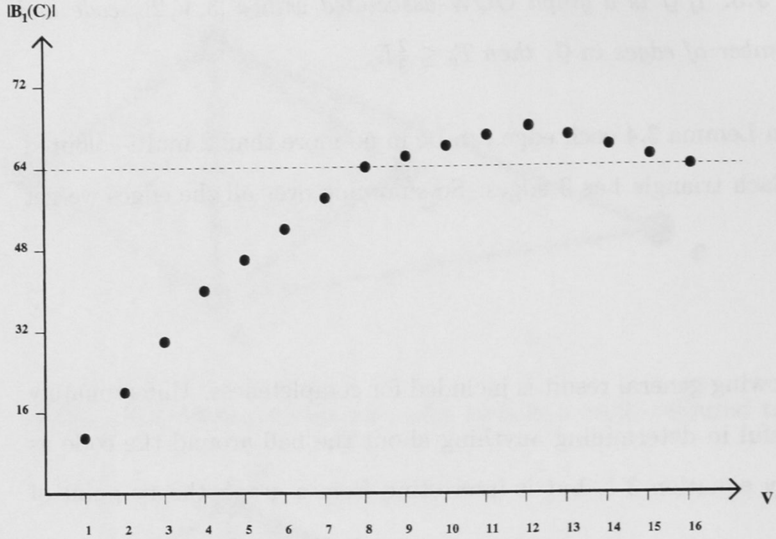

Figure 3.4: Equation 3.7 for $[3, V, 2]_4$ codes

Proof. From Corollary 3.5 and Theorem 3.3 we find that

$$|B_1(\mathcal{C})| \leq (1 + 3(q-1))V - 2E + \frac{2}{3}E \quad \text{and so}$$

$$(3.8) \quad |B_1(\mathcal{C})| \leq (1 + 3(q-1))V - \frac{4}{3}E.$$

We then use the result of Corollary 2.10, $E \geq \frac{q}{2}(z-1)z + wz$, to obtain equation 3.7.

♡

Corollary 3.8. *Let $\mathcal{C}$ be a $[3, V, 2]_q$ code such that $|B_1(\mathcal{C})| = p_q(3, V, 2, 1)$. Let $\mathcal{G}$ be the graph OQW-associated with $\mathcal{C}$. Then $\mathcal{G}$ will have the minimum number of edges.*

Proof. From equation 3.8 we see that adding edges to a graph can only reduce the size of the ball around the associated code. ♡

We plot equation 3.7 for $[3, V, 2]_4$ codes (Figure 3.4). The line at $B_1(\mathcal{C}) = 64$ shows the size of $H(3, 4)$. We can see that to obtain a covering code we need $8 \leq V \leq 16$.

To generalise this we solve equation 3.7, assuming that $w = 0$ to simplify things.

(3.9) $q^3 = (1 + 3(q-1))V - \frac{4}{3}\left(\frac{3q}{2}(z-1)z\right)$

(3.10) $0 = (V - q^2)(\frac{2V}{q} - q)$

When we solve equation 3.10 we find that for $V < \frac{1}{2}q^2$ it is not possible to find a $[3, V, 2]_q$ code $\mathcal{C}$ such that $|B_1(\mathcal{C})| = q^3$. (It is also not possible for $V > q^2$ but that is already ruled out by the Singleton bound.) So if we are searching for $[3, V, 2]_q 1$ codes we need $\frac{1}{2}q^2 \leq V \leq q^2$.

3.3 Codes which meet the Upper Bound

Equation 3.7 gives us a good bound on $|B_1(\mathcal{C})|$ for codes where $V \leq \frac{1}{2}q^2$. We know this bound can only be achieved if every edge in $\mathcal{G}$ is in two multi-coloured triangles and there is a minimum number of edges. We now look at what other properties the graph must have in order to meet bound 3.7.

Theorem 3.9. *If $\mathcal{G}$ is a graph OQW-associated with a $[3, V, 2]_q$ code, such that every edge is in 2 multi-coloured triangles then the following hold:*

1. *Each connected component of $\mathcal{G}$ is $3j-$regular, for some integer j*
2. *The number of vertices of each connected component of $\mathcal{G}$ must be exactly $(j+1)^2$.*
3. *Any two adjacent vertices of $\mathcal{G}$ are mutually adjacent to $(j+1)$ other vertices.*

Proof. 1. Looking at Figure 3.1, the central edge, ab, is in two multi-coloured triangles. For this to happen each of the vertices incident with that edge must have one of each colour edge.

If we add another black edge, ae, to vertex a then neither of the coloured edges already in the graph, ac or ad can be used to allow the new black edge

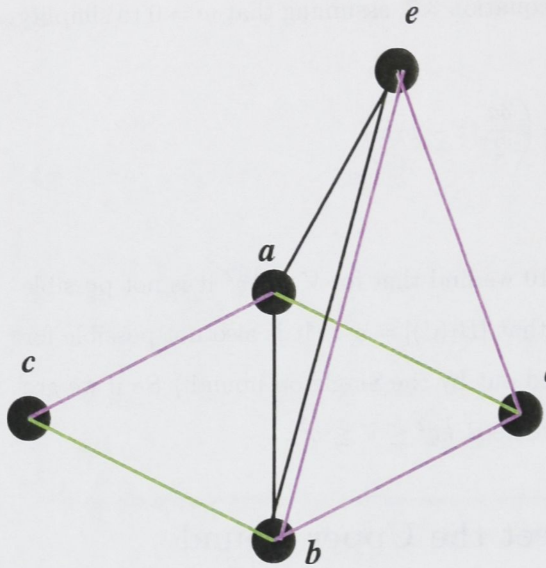

Figure 3.5: Edge ae cannot form a multi-coloured triangle with vertices c or d unless double edges are permitted.

to be in a multi-coloured triangle otherwise a double edge is required (figure 3.5). Thus more edges incident with vertex a must be added to enable ae to be in multi-coloured triangles. Thus we need each vertex to have the same number of incident edges of each colour.

If a vertex v has x edges of colour α incident with it, then every edge adjacent to v by an edge of colour α will also have x edges of colour α incident with it. Thus all vertices in any connected component must have an equal number of edges of each colour incident with them. There are 3 colours of edges, and hence each connected component of $\mathcal{G}$ must be $3j$-regular.

2. In part 1 we established that every vertex is incident with j edges of each colour. If we look at $\mathcal{G}_\alpha$, the single coloured sub-graph of $\mathcal{G}$, then each vertex is part of a complete graph on $(j+1)$ vertices. Each $K_{(j+1)}$ is disjoint, hence the number of vertices must be a multiple of $(j+1)$.

Each $\mathcal{G}_\alpha$ is a collection of disconnected $K_{j+1}s$. For a K_{j+1} of colour α to be able to exist in $\mathcal{G}$ there needs to be at least $j+1$ components to $\mathcal{G}_\alpha$.

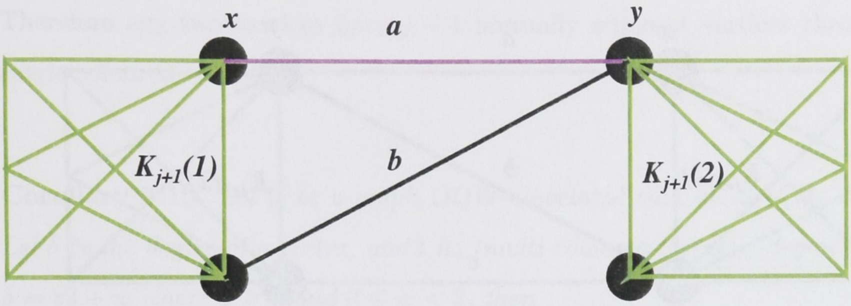

Figure 3.6: Edge a is between 2 vertices which are in different $(j+1)-$cliques of colour β.

Thus each $\mathcal{G}_\alpha$ must contain at least $j+1$ components; there must be at least $(j+1)^2$ vertices in each connected component of $\mathcal{G}$.

Choose an edge a between vertices x and y of colour α that is in a multi-coloured triangle. This edge connects 2 vertices which are in different $(j+1)-$cliques of colour j $(K_{J+1}1, K_{j+1}2)$, and is adjacent to an edge of colour γ which also connects $K_{j+1}(1)$ and $K_{j+1}(2)$ Eg Figure3.6.

To include edge a in another multi-coloured triangle requires another edge b of colour γ. To be in a triangle with edge a, edge b must include either vertex x or y. As all edges of colour β adjacent to a are in $K_{j+1}1$ or $K_{j+1}2$, edge b must include another vertex from either $K_{j+1}(1)$ or $K_{j+1}(2)$ to complete the multi-coloured triangle Eg Figure 3.7.

Thus every $j+1-$clique must be adjacent to every other $j+1-$clique; there must be $j+1$ $(j+1)-$cliques, n^2 vertices in all. Since each clique contains $j+1$ vertices, there must be $(j+1)^2$ vertices.

3. If two vertices a and b are adjacent, and we would like to count the number of mutually adjacent vertices, then we are counting the number of triangles which involve a and b. Every edge is in 2 multi-coloured triangles (including the one connecting our two vertices). Thus we have 2 mutually adjacent vertices through multi-coloured triangles. Every vertex has j edges of colour α. Thus any two vertices are in a single coloured $(j+1)$-clique.

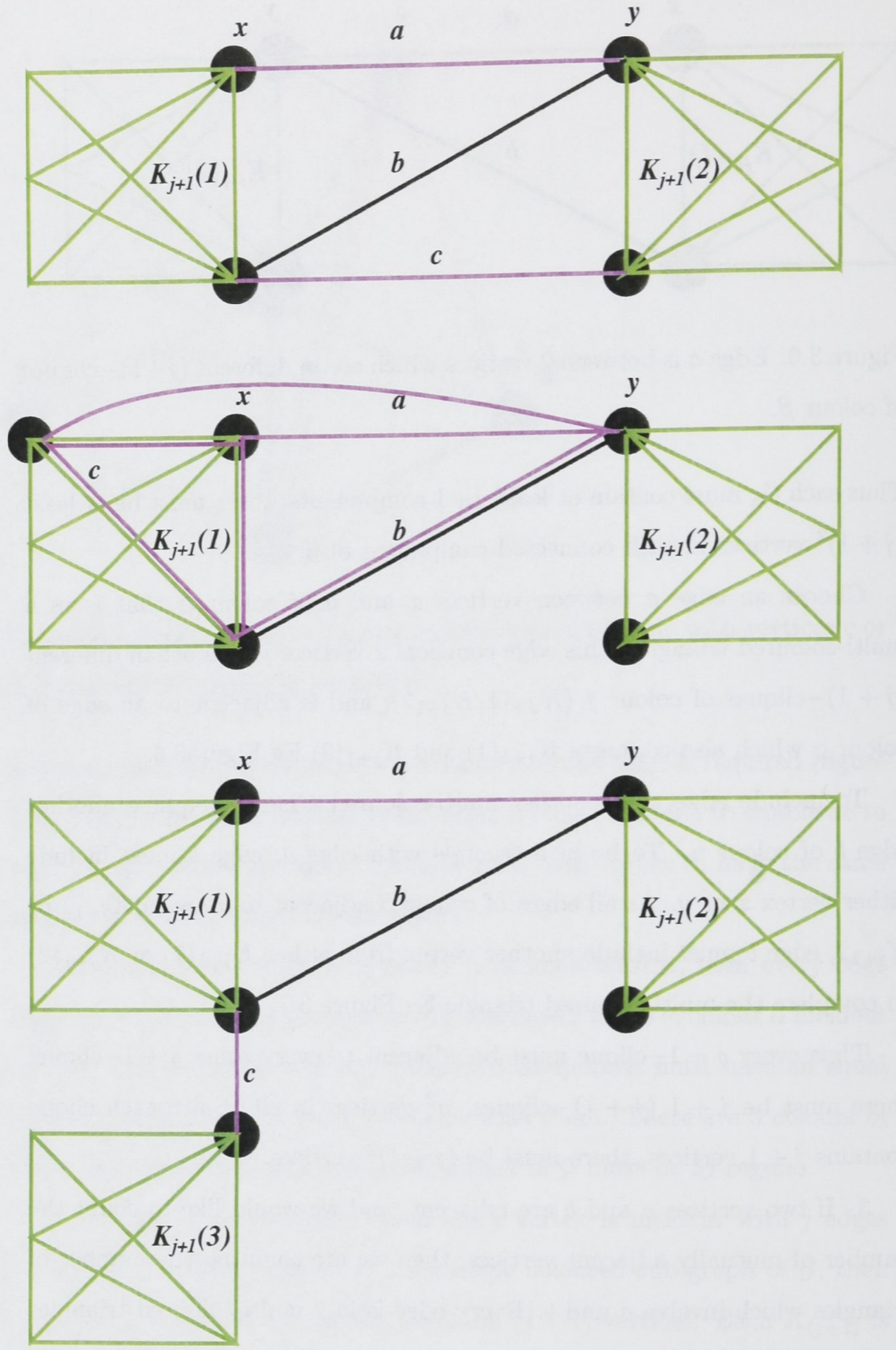

Figure 3.7: If edge b is in 2 mulit-coloured triangles then edge c must have vertices from $K_{j+1}(1)$ and $K_{j+1}(2)$.

Therefore any two vertices have $j-1$ mutually adjacent vertices through single coloured triangles.

$\heartsuit$

Corollary 3.10. *Let $\mathcal{G}$ be a graph OQW-associated with an $[3,V,2]_q$ code. Let δ be the degree of a vertex, and t its (multi-coloured triangle)-degree. Let $\delta = z3 + w$ where $z \geq 1$ and $0 \leq w < 3$, then*

$$t \leq z + \frac{w}{2} \tag{3.11}$$

Proof. From Theorem 3.9 part 1, the maximum number of multi-coloured triangles can only be achieved if there is an equal number of edges of each colour at each vertex. Thus the number of multi-coloured triangles is bounded by the number of groups of 3 different coloured edges, *i.e.* z. The other edges can only make triangles amongst themselves. Thus if there is 1 ungrouped edge, it cannot be in any multi-coloured triangles, and if there are 2 ungrouped edges they can form a triangle, and each be in one multi-coloured triangle.

$\heartsuit$

The number of groups of 3 different coloured edges incident with a vertex is bounded by the minimum number of edges of a single colour. And so we get the following result.

Corollary 3.11. *Let $\mathcal{G}$ be a graph OQW-associated to an $[3,V,2]_q$ code. Let δ_i be the i^{th} coloured degree of a vertex. Let $x = \min\{\delta_1, \delta_2, \delta_3\}$. Then*

$$t \leq 3x + \left\lfloor \frac{d-3x}{2} \right\rfloor \tag{3.12}$$

For example in figure 3.8, surrounding vertex a we have a group of 3 edges of different colours, ab, ac, ad to which we can add sufficient edges so that each are in 2 multi-coloured triangles. The 2 edges left, ae and af, can only be in 1 multi-coloured triangle. Thus if a vertex has degree 5 it can be in at most 4 multi-coloured triangles.

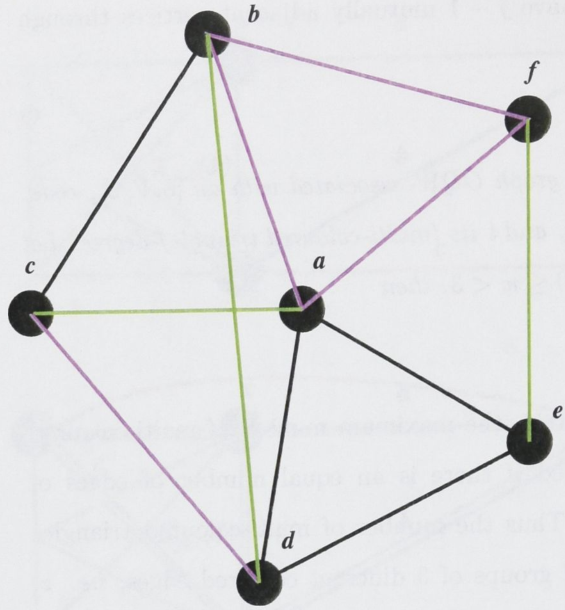

Figure 3.8: Vertex a has degree 5, is incident with only 4 multi-coloured triangles.

Lemma 3.12. *Let $\mathcal{G}$ be a graph OQW-associated to an $[3, V, 2]_q$ code such that vertex v has colour degree 1 for each colour. Then for v to be in 3 multi-coloured triangles, v must be in a multi-coloured 4-clique.*

Proof. From Lemma 2.12 we know that $\binom{3}{2} = 3$ is the maximum number of triangles that v can be incident with, and this can occur if and only if v is in a 4-clique.

We know it is possible to colour a 4 clique so that v is in 3 multi-coloured triangles. ♡

If $j \geq 2$ then there need not be a multi-coloured 4-clique for every edge to be in 2 multi-coloured triangles as figure 3.9 shows.

Combining Theorem 3.9 with Theorem 2.13 we find that if $q = 4$ then there are 6 known graphs (including the trivial solution) which satisfy all the requirements of Theorem 2.6 and have every edge in 2 multi-coloured

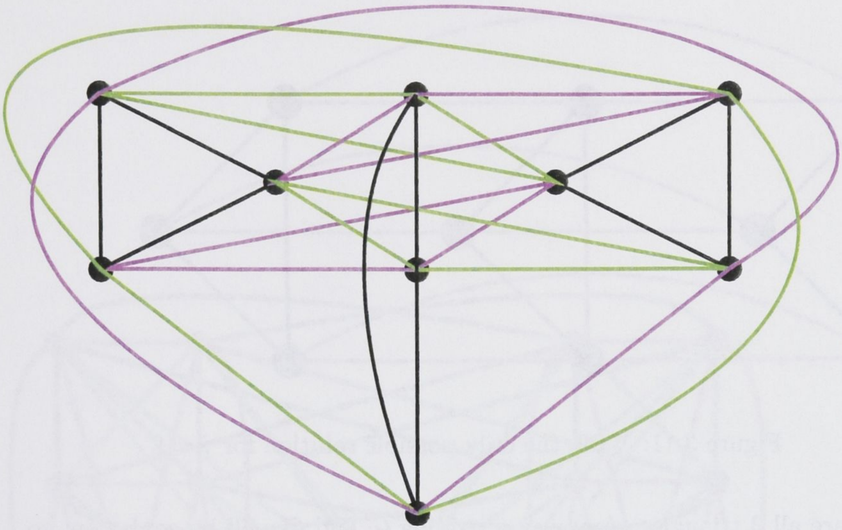

Figure 3.9: Every edge is in 2 multi-coloured triangles

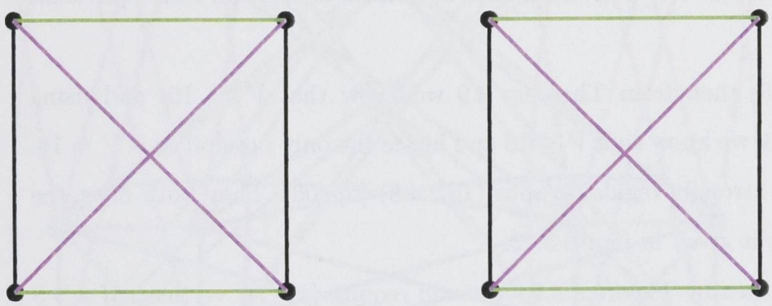

Figure 3.10: Two disjoint K_4s can satisfy equation 3.7

triangles, and 3 known graphs which in addition satisfy bound 3.7.

If $j = 0$ then we have the trivial solution.

If $j = 1$ then we have K_4 or a disjoint collection of multi-coloured K_4s. If we have more that 8 vertices, then there is not enough edges of each colour to satisfy corollary 2.10. Thus we have K_4 or 2 disjoint K_4s as in figure 3.10.

If $j = 2$ then, from Theorem 3.9 part 2 we know that $V = 9$. The (9,6,3,6) strongly regular graph displayed in figure 3.9 meets the criteria. This graph is the only possibility. This can be seen by looking at each $\mathcal{G}_i$,

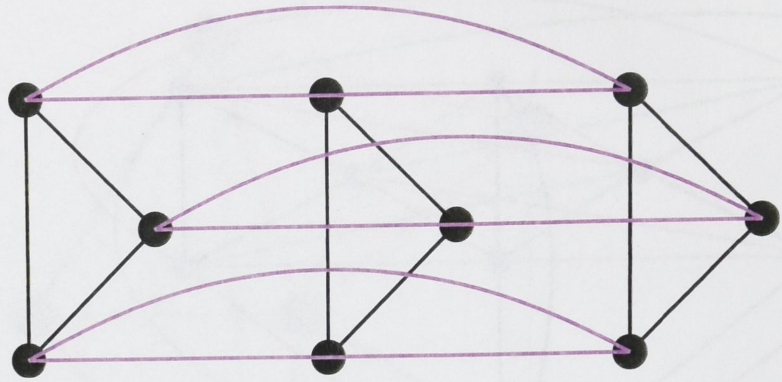

Figure 3.11: V=9, the only possible solution for $j = 2$

they are all 3 triangles. Any way of adding $\mathcal{G}_i$ and $\mathcal{G}_j$ will be equivalent to figure 3.11. Let $a := 000$ then we can choose either $e = 111$, $i = 222$ or $h = 111$, $f = 222$. However, both form equivalent codes, and thus equivalent graphs.

If $j = 3$, then from Theorem 3.9 we know that $V = 16$, and using Corollary 2.9 we know that $V \leq 16$ and hence the only possibility is $V = 16$. There are 2 strongly regular graphs (16,9,4,6) [Spe06]. They both meet the bound. One is given in figure 3.12.

If $j > 3$ then by Theorem 3.9 we would require $(j + 1)^2$ vertices. If $j > 3$ then $(j + 1)^2 > 16$, which contradicts Corollary 2.9.

The only examples we have of graphs for which every edge is in 2 multi-coloured triangles are strongly regular. However we do not have a result as yet which determines the number of mutually adjacent vertices to a pair of non-adjacent vertices. If this were the case we could restrict our search to strongly regular graphs, which for $V \leq 36$ are completely catalogued [Spe06].

The only examples that also meet bound 3.7 are the strongly regular (16,9,4,6) graphs, and 2 multi-coloured K_4 (figure 3.10) .

For $q = 4$ we know these are the only graphs which meet the bound.

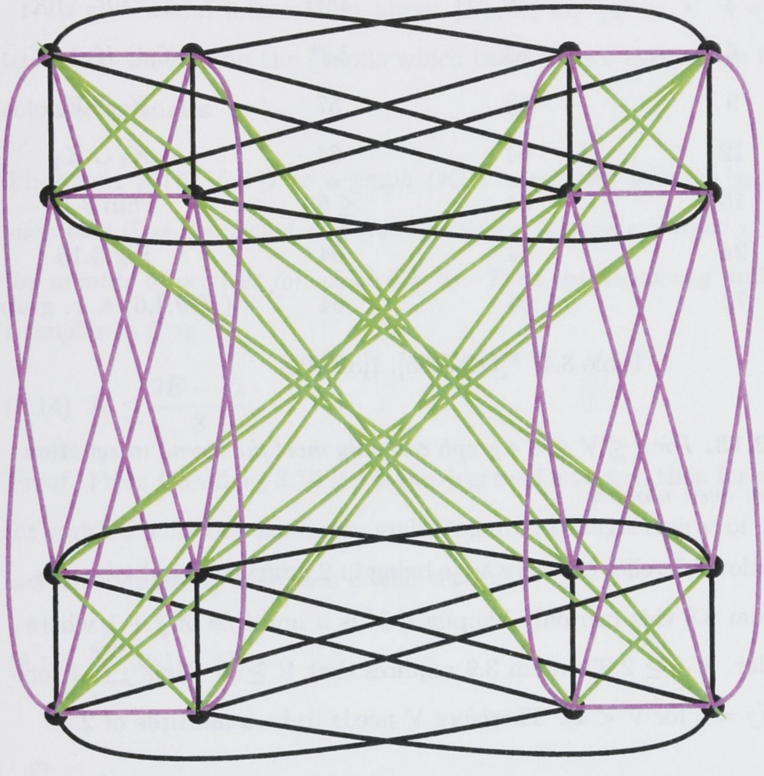

Figure 3.12: A $(16, 9, 4, 6)$ strongly regular graph coloured to obey the OQW-association method.

V	min E (eqn2.10)	Bound (eqn3.7)	$p_4(3, V, 2, 1)$	graph
1	0	10	10*	K_1
2	0	20	20*	$K_1 \cup K_1$
3	0	30	30*	$K_1 \cup K_1 \cup K_1$
4	0	40	40*	$K_1 \cup K_1 \cup K_1 \cup K_1$
5	3	46	45*†	$K_3 \cup K_1 \cup K_1$
6	6	52	52*	$K_4 \cup K_1 \cup K_1$
7	9	58	57	$K_4 \cup K_3$
8	12	64	64	$K_4 \cup K_4$
9	18	64	≤ 63	Thm 3.16
10	24	64	64	Fig 3.16
16	72	64	64	(16,9,4,6) s. r. graph.

Table 3.1: *[OQW05], †[SK69b]

Corollary 3.13. *For $4 \leq V \leq 8$ a graph can only meet the bound in equation 3.7 if V is an even number.*

Proof. Equation 3.7 relies on every edge being in 2 multi-coloured triangles. From Theorem 3.9 this can only happen if V is a multiple of $j + 1$ where $\mathcal{G}$ is $3j$-regular. If $j \geq 2$ Theorem 3.9 requires that $V \geq 9$. Thus $j = 1$ for $4 \leq V \leq 8$. ($j = 0$ for $V < 4$). Therefore V needs to be a multiple of 2. ♡

From Corollary 3.13, for those values of V which are odd, the value of the bound in equation 3.7 must be reduced by at least 1. There is at least one graph (see table 3.1) which reaches this bound in all cases where equation 3.7 is useful (*i.e.* $V \leq 8$ or $V = 16$). This gives the true maximum value in the case $q = 4$ for all values of V where $0 \geq V \geq 8$ or $V = 16$. For $9 \leq V \leq 15$ eqn 3.7 is not useful. The cases of $V = 9$ and $V = 10$ are considered later in the chapter.

Theorem 3.14. *The minimum size of a $[3, V, 2]_4 1$ code is 8.*

(3.13) $K_4(3, 2, 1) = 8$

Proof. From equation 3.7 we know that $K_4(3, 2, 1) \geq 8$ and figure 3.10 gives an example of an OQW-associated graph that achieves this bound. ♡

To give useful information about $|B_1(\mathcal{C})|$ for $\frac{1}{2}q^2 < V < q^2$ we need to look at bounds on the graphs which cannot have every edge in 2 multi-coloured triangles.

Theorem 3.15. *Let $\mathcal{G}$ be a graph OQW-associated with an $[n, V, d]_q$ code such that $\mathcal{G}$ is connected and not $3j$-regular for some integer j. Let V_δ be the number of vertices for which $\delta \neq 3j$. Then the number of multi-coloured triangles in $\mathcal{G}$ is*

(3.14) $T_c \leq \dfrac{2E - V_\delta}{3}$

Proof. From Corollary 3.10 if δ is the degree of a vertex, then for each vertex for which $\delta \neq 3j$ the maximum multi-coloured triangle degree of that vertex is $\delta - 1$. Summing over each of the vertices

(3.15) $\displaystyle\sum_{i=1}^{V} \delta(i) = 2E,$

(3.16) $\displaystyle\sum_{i=1}^{V} \delta(i) - 1 = 2E - V_\delta$

Each triangle contains 3 vertices and so

(3.17) $T_c \leq \dfrac{2E - V_\delta}{3}.$

♡

This result has implications for deducing the bound in equation 3.7, when the minimum number of edges is not a multiple of 3.

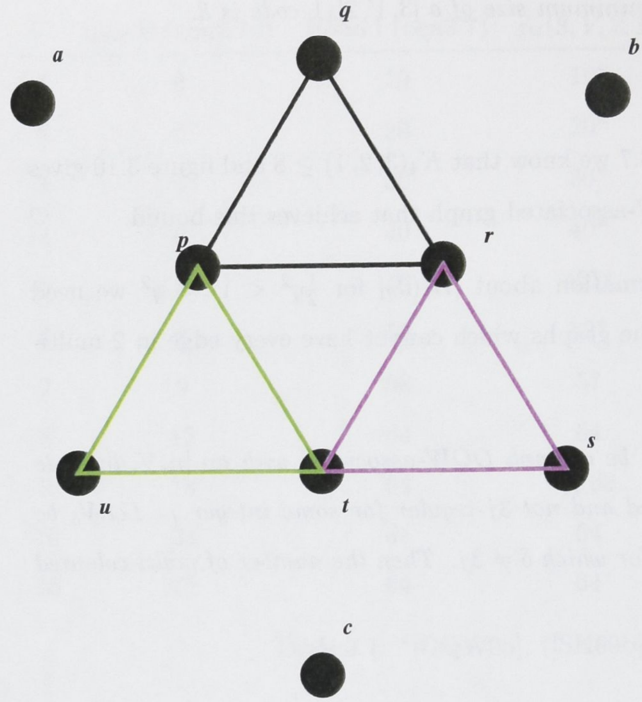

Figure 3.13: Vertices a, b and c can have degree $\delta = 3$

Theorem 3.16.

(3.18) $p_4(3, 9, 2, 1) \leq 63$

Proof. Let $\mathcal{C}$ be a $[3, 9, 2]_4$ code with graph $\mathcal{G}$ OQW-associated. Using Corollary 2.11 we know that $\mathcal{G}$ has at least 6 edges of each colour, with at least one single coloured triangle of each colour. Therefore $\mathcal{G}$ has a maximum of 3 vertices with degree 3 or 6. Thus $V_\delta \geq 6$

Using Theorem 3.3 and Theorem 3.15 we find that

(3.19) $|B_1(\mathcal{C})| \leq 90 - 2 \times 18 + \dfrac{2 \times 18 - 6}{3} = 64$

So if we are to find a covering code we must have 3 vertices of degree 3 or 6. There are exactly 2 arrangements of the 3 single coloured triangles which can achieve this (Figures 3.13 and 3.14).

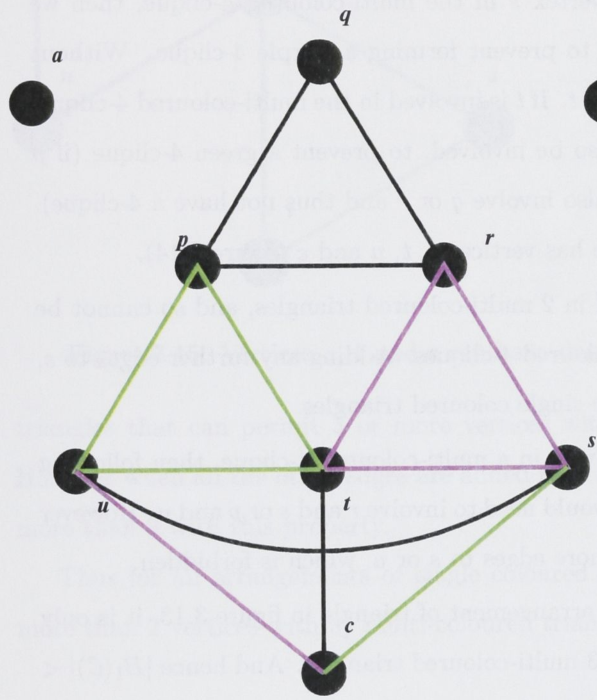

Figure 3.14: Vertices s, t, u and c form a multi-coloured 4-clique.

From Lemma 3.12 we know that if a vertex v has degree 3, then for every edge incident with v to be in 2 multi-coloured triangles, we need v to be part of a multi-coloured 4-clique.

Thus in figure 3.13 we need vertices a, b and c to be part of multi-coloured 4-cliques. As we have only 3 vertices, the 4-clique must involve at least one vertex that has degree 2 for some colour.

If we want to involve vertex s in the multi-coloured 4-clique, then we must involve vertex t or r to prevent forming a purple 4-clique. Without loss of generality let's choose t. If t is involved in the multi-coloured 4-clique, then vertex u or p must also be involved, to prevent a green 4-clique (if p is involved, then we must also involve q or r and thus not have a 4-clique). Our multi-coloured 4 clique has vertices s, t, u and c (figure 3.14).

The edges added are all in 2 multi-coloured triangles, and so cannot be part of any further multi-coloured 4-cliques. Adding any further edges to s, t, u, or c would create more single coloured triangles.

If we would like to involve q in a multi-coloured 4-clique, then following the same logic as above we would need to involve r and s or p and u. However this would require adding more edges to s or u, which is forbidden.

In conclusion, using the arrangement of triangle in figure 3.13, it is only possible to have 1 vertex in 3 multi-coloured triangles. And hence $|B_1(\mathcal{C})| < 64$.

For Figure 3.15, we need vertex c to be in 6 multi-coloured triangles. The only way to do this is use 2 edges of colour i to connect the vertices of the triangles not of colour i, for each colour. There are 2 different arrangements. Importantly neither arrangement uses vertices a or b. This leaves one edge of each colour to be added to the graph, thus at most one of a and b can have degree 3. Therefore at most 2 vertices can have $3j$ multi-coloured triangles. Hence with this arrangement of single coloured triangles $|B_1(\mathcal{C})| < 64$.

Figures 3.13 and 3.15 are the only arrangement of the three single coloured

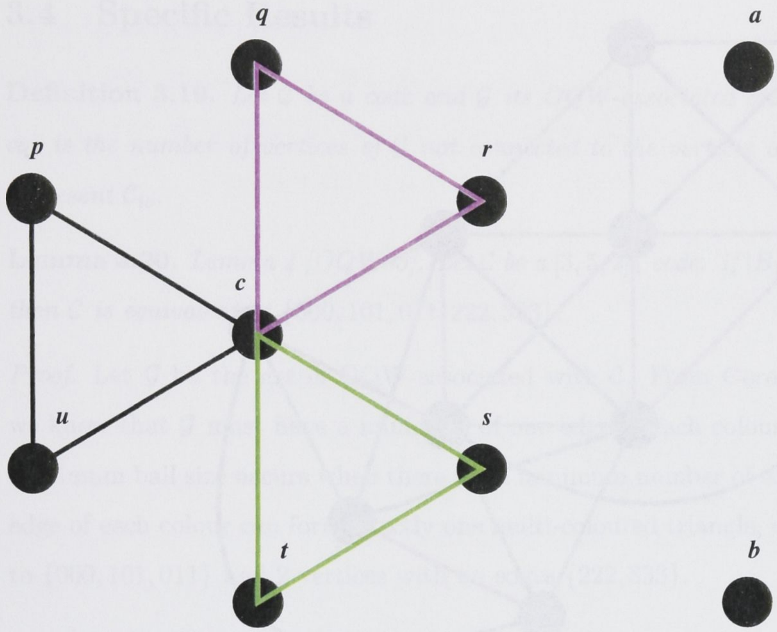

Figure 3.15: Vertices a, b and c can have degree $\delta = 3j$ for some j

triangles that can permit 3 or more vertices with degree a multiple of 3. However when all the other edges are added into the graph, there can be no more than 2 with this property.

Thus for all arrangements of single coloured triangles there can be no more than 2 vertices with $3j$ multi-coloured triangles. Thus $V_\delta \geq 7$

(3.20) $|B_1(\mathcal{C})| \leq 90 - 2 \times 18 + \dfrac{2 \times 18 - 7}{3} = 63\dfrac{2}{3}$

There must be at least one word not covered by $B_1(\mathcal{C})$.

♡

Given that $|H(3,4)| = 64$ we get the following result.

Corollary 3.17. *There is no* $[3,9,2]_4 1$ *code.*

Theorem 3.18.

(3.21) $p_4(3,10,2,1) = 64$

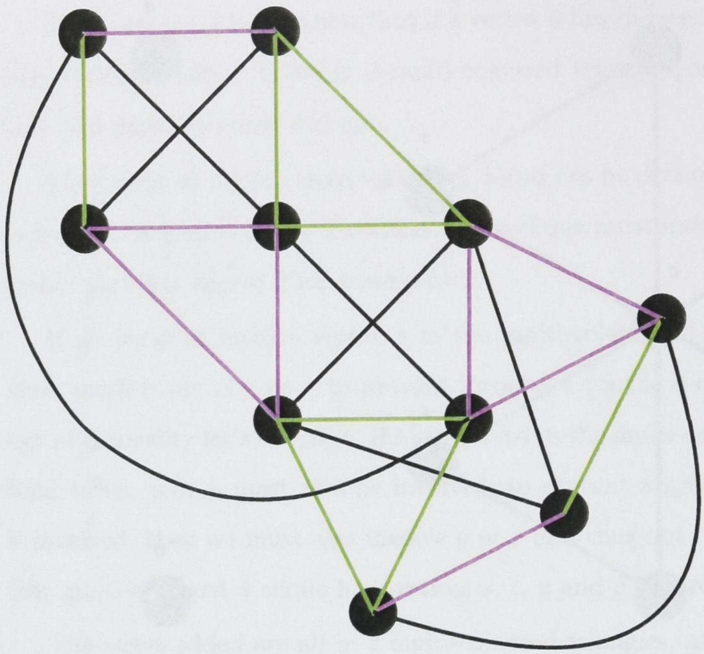

Figure 3.16: A graph OQW-associated with an $[3, 10, 1]_4 1$ code

Proof. Because $|H_{3,4}| = 64$ we know that $p_4(3, 10, 2, 1) \leq 64$. This can be satisfied by the code OQW-associated with following graph (figure 3.16). It has $8 \times 3 = 24$ edges and 12 multi-coloured triangles.

(3.22) $|B_1(\mathcal{C})| = 100 - 2 \times 24 + 12 = 64$

♡

Figure 3.16 is not the only $[3, 10, 2]_4 1$ code, just one example to show that a covering code can be found.

For $V \in \{11, 12, 13, 14, 15\}$ it is still open as the whether a $[3, V, 2]_4 1$ code exists.

3.4 Specific Results

Definition 3.19. *Let $\mathcal{C}$ be a code and $\mathcal{G}$ its OQW-associated graph. Then α_{is} is the number of vertices of $\mathcal{G}$ not connected to the vertices of $\mathcal{G}$ which represent $\mathcal{C}_{is}$.*

Lemma 3.20. *Lemma 4 [OQW05]. Let $\mathcal{C}$ be a $[3,5,2]_4$ code. If $|B_1(\mathcal{C})| = 45$ then $\mathcal{C}$ is equivalent to $\{000, 101, 011, 222, 333\}$.*

Proof. Let $\mathcal{G}$ be the graph OQW-associated with $\mathcal{C}$. From Corollary 2.10 we know that $\mathcal{G}$ must have a minimum of one edge of each colour, and the maximum ball size occurs when there are a minimum number of edges. One edge of each colour can form exactly one multi-coloured triangle, equivalent to $\{000, 101, 011\}$ and 2 vertices with no edges $\{222, 333\}$. $\heartsuit$

Lemma 3.21. *Let $\mathcal{G}$ be the graph OQW-associated with $\mathcal{C}$ a $[3,V,2]_4$ code. Then*

(3.23) $|H_{is} \cap B_1(\mathcal{C})| = 8|\mathcal{C}_{is}| - |\mathcal{C}_{is}|^2 + \alpha_{is}.$

Proof. $\mathcal{C}_{is}$ is equivalent to the s^{th} component of $\mathcal{G}_i$. For each word x with an s in the i^{th} position $|H_{is} \cap B_1(x)| = 7$. If there are two words, x and y, in $\mathcal{C}_{is}$ then by the principle of inclusion exclusion

(3.24) $|H_{is} \cap B_1(\mathcal{C}_{is})| = |H_{is} \cap B_1(x)| + |H_{is} \cap B_1(y)| - |H_{is} \cap B_1(x) \cap B_1(y)|$

Since $x, y \in \mathcal{C}_{is}$, they are connected by an edge, and hence $|H_{is} \cap B_1(x) \cap B_1(y)| = 2$. Each word in H_{is} can only be in the ball of 2 codewords. Thus $|B_1(x) \cap B_1(y) \cap B_1(z)| = 0$. Thus the number of words which are already in some other ball is $2\binom{|\mathcal{C}_{is}|}{2} = |\mathcal{C}_{is}|(|\mathcal{C}_{is}| - 1)$.

(3.25) $|H_{is} \cap B_1(\mathcal{C}_{is})| = 7|\mathcal{C}_{is}| - |\mathcal{C}_{is}|(|\mathcal{C}_{is}| - 1) = 8|\mathcal{C}_{is}| - |\mathcal{C}_{is}|^2$

We are looking for $|H_{is} \cap B_1(\mathcal{C})|$. So we must include the contribution from the other codewords.

Let $v \in \mathcal{C} \setminus \mathcal{C}_{is}$ then $|B_1(v) \cap H_{is}| = 1$ If v is adjacent to any member of $\mathcal{C}_{is}$ then $B_1(v) \cap H_{is} \subset B_1(\mathcal{C}_{is})$, and has already been counted. But if v is not adjacent by any colour edge to any member of $\mathcal{C}_{is}$ then $B_1(v) \cap H_{is}$ is different and must be counted. There are α_{is} such vertices.

♡

Theorem 3.22. *Let $\mathcal{C}$ be a $[3, V, 2]_4$ code. Then*

(3.26) $K_{is} = 16 - 7|\mathcal{C}_{is}| + |\mathcal{C}_{is}|^2 - \alpha_{is}$.

Proof. From Definition 1.12

$$K_{is} = |H_{is} \setminus B_1(\mathcal{C})| + |\mathcal{C}_{is}|.$$

Since $|H_{is}| = 16$

$$|H_{is} \setminus B_1(\mathcal{C})| = 16 - |H_{is} \cap B_1(\mathcal{C})|,$$

and so

(3.27) $K_{is} = 16 - |H_{is} \cap B_1(\mathcal{C})| + |\mathcal{C}_{is}|$.

Then from Lemma 3.21

(3.28) $K_{is} = 16 - (8|\mathcal{C}_{is}| - |\mathcal{C}_{is}|^2 + \alpha_{is}) + |C_{is}|$

♡

If $\mathcal{C}$ is a $[3, 6, 2]_4$ code then table 3.2 displays possible values for K_{is}.

Lemma 3.23. *Lemma 5[OQW05]. Let $\mathcal{C}$ be a $[3, 6, 2]_4$ code.*

1. *If $|B_1(\mathcal{C})| = 52$ then $\mathcal{C}$ is equivalent to $\{000, 011, 101, 110, 222, 333\}$ and $K_{is} \leq 5$ for all i, s.*

2. *If $|B_1(\mathcal{C})| < 52$ and $K_{is} \leq 6$ for all i, s, then $\mathcal{C}$ is equivalent to $\{000, 011, 101, 123, 230, 312\}$ with $|B_1(\mathcal{C})| = 49$*

C_{is}	$\min K_{is}$	$\max K_{is}$
0	10	10
1	5	10
2	2	6
3	1	4

Table 3.2: Possible values for K_{is} for a $[3,6,2]_q$ code

Proof. Part 1. From Corollary 2.10 we know that there must be at least 2 edges of each colour. By Theorem 3.3 if $|B_1(\mathcal{C})| = 52$ there must be 4 multi-coloured triangles. Using Corollary 3.5 means that every edge must be in 2 multi-coloured triangles. From Lemma 3.12 we know these edges must form a multi-coloured K_4, equivalent to $\{000, 011, 101, 110\}$ and two vertices with no edges $\{222, 333\}$. To see that $K_{is} \leq 5$, see that $\mathcal{C}_{i0} = \mathcal{C}_{i1} = 2$ with 2 vertices not connected thus $K_{i0} = K_{i1} = 4$; $\mathcal{C}_{i2} = \mathcal{C}_{i3} = 1$ with 5 vertices not connected thus $K_{i2} = K_{i3} = 5$.

Part 2. If there exist i and s such that $|C_{is}| = 0$ then from table 3.2 $K_{is} = 10$.

If $|\mathcal{C}_{is}| = 3$ then $\mathcal{G}$ has a single coloured triangle and there are at least 2 more edges of colors 2 and 3. Only one end of each of those edges can be incident with the single coloured triangles, the 3 vertices not in the single coloured triangle must have at least 4 edge ends between them. Hence one of them must be adjacent to at least 2 vertices. Using Theorem 3.22 we conclude that if $|\mathcal{C}_{is}| = 3$ then for some $t \neq s$ $K_{it} \geq 7$.

Let $|\mathcal{C}_{is}| \in \{1, 2\}$. There will be 2 edges of each colour, 6 edges in all. Every triangle is multi-coloured ($|\mathcal{C}_{is}| = 3$ is required for a single coloured triangle). Each vertex can be incident with at most one edge of each colour, so the maximum degree of the vertex is 3. If there is a codeword v of degree 2, then there is one colour of edge that is not adjacent to v, and so for some i, s, $\mathcal{C}_{is} = \{v\}$, and hence $K_{is} = 7$.

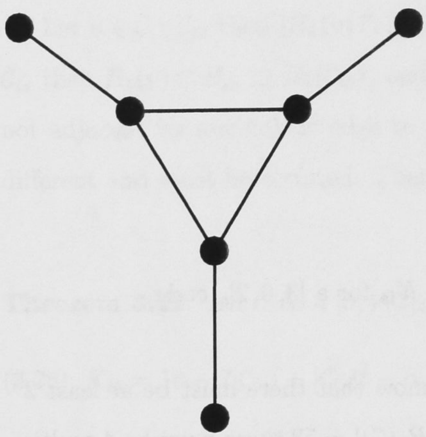

Figure 3.17: A $[3, 6, 2]_4$ code where $B_1(\mathcal{C}) < 52$ and $K_{is} \leq 6$ $\forall i, s$.

Thus if $\mathcal{C}$ is a $[3, 6, 2]_4$ code with $B_1(\mathcal{C}) < 52$ and $K_{is} \leq 6$ for all i, s then the degree of every codeword is 1 or 3. There are $6 \times 2 = 12$ edge ends, thus we must have 3 vertices of degree 3, and 3 vertices of degree 1. The only graph which satisfies this is given in Figure 3.17. This can be coloured in only one way to avoid a single coloured triangle.

♡

Chapter 4

$[n, V, 2]_q$ Codes

The association methods defined in Chapter 2 and developed in Chapter 3 can be extended to general codes. Many of the ideas and results of Chapter 3 can be extended, perhaps in a weaker, form to general codes. We also look at general coding ideas, not previously discussed for the specific case of $n = 3$.

Our goal is to find techniques to determine $K_q(n, 2, 1)$, we begin by evaluating the difficulties of the problem.

Lemma 4.1. *[OQW05] Let $\mathcal{C}$ be an $[n, V, d]_q$ code. Then there exist $[n, V', d]_q$ codes for any $V' \leq V$.*

Lemma 4.2. *[OQW05] Let $\mathcal{C}$ be an $[n, V]_q R$ code. Then there exists $[n, V']_q R$ codes for any $V' \geq V$.*

However for $V_1 < V_2 < V_3$ the existence of $[n, V_1, d]_q R$ and $[n, V_3, d]_q R$ codes does not imply the existence of an $[n, V_2, d]_q R$ code.

For example:

In Table 3.1 we found that a $[3, 8, 2]_4 1$ code exists, then in Theorem 3.18 we found that a $[3, 10, 2]_4 1$ code exists. But from Corollary 3.17 we found that no $[3, 9, 2]_4 1$ code exists.

For an even smaller example [OQW05] take the $[3, 2, 2]_2 1$ Hamming code: this is also a $[3, 2, 3]_2 1$ code. If we take all words of length 2, then add a parity check bit then we obtain the unique $[3, 4, 2]_2 1$ code. However the only $[3, 3, 2]_2$ code $\{(000), (011), (110)\}$ has a covering radius of 2.

This means that for a given n, d, q and r, to find the minimum V for which a $[n, V, d]_q R$ exists we must find a general bound, or show the non-existence of every smaller case.

Thus if we were only looking for codes with a certain minimum distance, or a certain covering radius, then the problem would be much simpler. Those two problems are well studied.

4.1 Generalising the Ideas of Chapter 3

We would like a generalized version of Theorem 3.4, but we need a broader definition of edge and multi-coloured triangle.

Definition 4.3. *An x-edge is a set of x edges that are between the same pair of vertices.*

A 2-edge will be called a double edge.

Definition 4.4. *Let $\mathcal{G}$ be a multi-graph with a maximum of n edges between each vertex, any two edges between two vertices must be different colours. Then a generalized multi-coloured triangle is a triangle with n colours of edges, and any two vertices in the triangle having $n - 2$ edges between them.*

Figure 4.1 shows a generalized triangle for $n = 4$. Because there are only n colours, and $n - 2$ must be between any pair of vertices, a generalized multi-coloured triangle must contain $n - 3$ single coloured (single edged) triangles and one (single edged) multi-coloured triangle. To see this draw a multi-coloured triangle. We need to add $n - 3$ edges between each vertex. Choose two vertices of the multi-coloured triangle. To add another edge

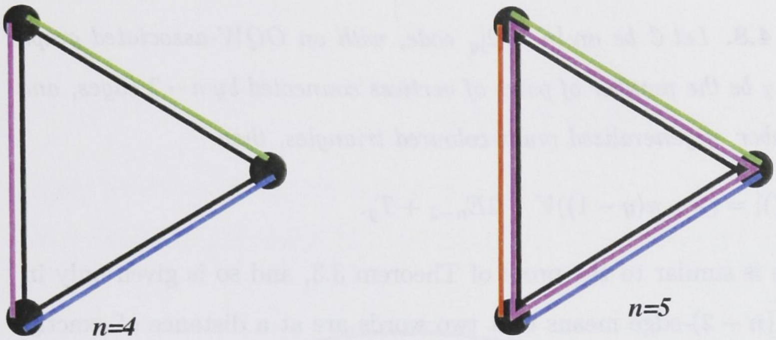

Figure 4.1: Generalized multi-coloured triangles, $n = 4$ and $n = 5$

between those vertices we can't use any colours that are already in the triangle, hence we can only use the $n - 3$ colours not already used. This applies to each of the edges of the multi-coloured triangle.

Thus we can also describe a generalized multi-coloured triangle as a (single edged) multi-coloured triangle coupled with $n - 3$ single coloured (single edged) triangles.

Definition 4.5. *The single coloured triangles in a generalized multi-coloured triangle will be called the* base *of the generalized multi-coloured triangle*

Definition 4.6. *3 vertices which are connected by single coloured triangles of x colours will be called an x-single coloured triangle.*

If there are 2 single coloured triangles connecting the same 3 vertices, this will be called a double single edged triangle.

Definition 4.7. *Let $\mathcal{G}$ be a graph OQW-associated to an $[n, k, d]_q$ code. Then $\mathcal{H}_{is}$ is the subgraph created by taking the vertices associated with $\mathcal{C}_{is}$ and all edges joining any of those vertices.*

$\mathcal{H}_{is}$ is a generalization of $\mathcal{G}_{is}$ for multi-graphs. Thus for $n = 3$ $\mathcal{H}_{is} \equiv \mathcal{G}_{is}$. $\mathcal{H}_i = \bigcup_s \mathcal{H}_{is}$, thus $\mathcal{H}_i$ is the natural generalization of $\mathcal{G}_i$.

We can generalize Theorem 3.3 to make use of generalized multi-coloured triangles.

Theorem 4.8. *Let $\mathcal{C}$ be an $[n, V, 2]_q$ code, with an OQW-associated graph $\mathcal{G}$. Let E_{n-2} be the number of pairs of vertices connected by $n-2$ edges, and T_g the number of generalized multi-coloured triangles, then*

$$|B_1(\mathcal{C})| = (1 + n(q-1))V - 2E_{n-2} + T_g. \tag{4.1}$$

Proof. This is similar to the proof of Theorem 3.3, and so is given only in brief. Any $(n-2)$-edge means that two words are at a distance of exactly 2, and hence will have overlapping balls of radius 1. Hence any $n-2$ edge will reduce the size of $B_1(\mathcal{C})$ by 2. If there is a generalized multi-coloured triangle, then a word will have been excluded twice, and so by inclusion exclusion, we must add it again.

♡

As with Theorem 3.3, counting multi-coloured triangles is crucial. Counting multi-edges is more difficult than counting edges and is not approached in a general way.

Lemma 4.9. *If $\mathcal{C}$ is a $[4, V, 2]_q$ code with $\mathcal{G}$ its OQW-associated graph, then each double edge can be in at most 2 generalized multi-coloured triangles that contain the same coloured base.*

Proof. In the proof of Theorem 3.4 we were restricted by only having a single edge between any two vertices. Now we are allowed 2 edges between any two vertices.

However as shown in figure 4.2, there can be no more than 2 generalized multi-coloured triangles with the same coloured base.

Double edge ab is in two generalized multi-coloured triangles with vertices c and d. When we add a third generalized multi-coloured triangle with a black base, vertex e, we have to add sufficient black edges to form a complete graph. The extra 2 edges required to from purple and green single coloured triangles mean that we need a triple edge. This is disallowed by Theorem 2.14.

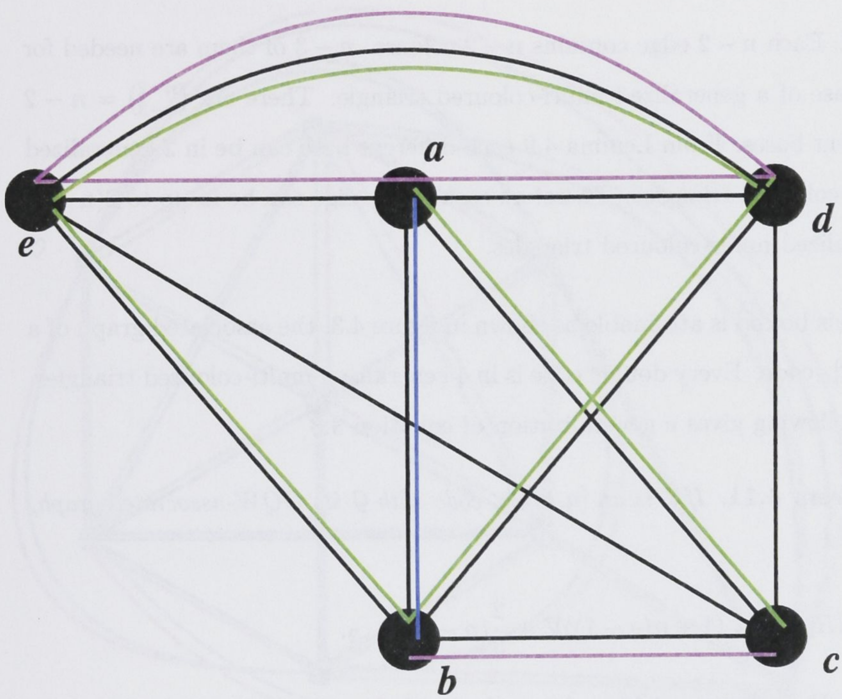

Figure 4.2: No more than 2 generalized multi-coloured triangles on the same base.

♡

This result can be further generalized.

Lemma 4.10. *If $\mathcal{C}$ is an $[n, V, 2]_q$ code with $\mathcal{G}$ its OQW-associated graph, then each $(n-2)$ edge can be in at most $2(n-2)$ generalized multi-coloured triangles.*

Proof. Each $n-2$ edge contains $n-2$ colours. $n-3$ of them are needed for the base of a generalized multi-coloured triangle. There are $\binom{n-2}{n-3} = n-2$ different bases. From Lemma 4.9 each different base can be in 2 generalized multi-coloured triangles. Therefore each $n-2$ edge can be in up to $2(n-2)$ generalized multi-coloured triangles. ♡

This bound is attainable as shown in figure 4.3, the associated graph of a $[4, 8, 2]_4$ code. Every double edge is in 4 generalized multi-coloured triangles. The following gives a generalization of equation 3.8

Theorem 4.11. *If $\mathcal{C}$ is an $[n, V, 2]_q$ code with $\mathcal{G}$ its OQW-associated graph, then*

$$(4.2)\quad |B_1(\mathcal{C})| \leq (1 + n(q-1))V + \frac{2}{3}(n-5)E_{n-2}.$$

Proof. From Lemma 4.10 we know that each $(n-2)$ edge can be in $2(n-2)$ generalized multi-coloured triangles. Therefore $\mathcal{G}$ can have at most $\frac{2}{3}(n-2)E_{n-2}$ multi-coloured triangles. Using Theorem 3.4 we get

$$(4.3)\quad |B_1(\mathcal{C})| \leq (1 + n(q-1))V - 2E_{n-2} + \frac{2}{3}(n-2)E_{n-2}.$$

♡

With this result, we can now use graphs to solve a larger set of problems. These will be used for the case $n = 4$ in chapter 5.

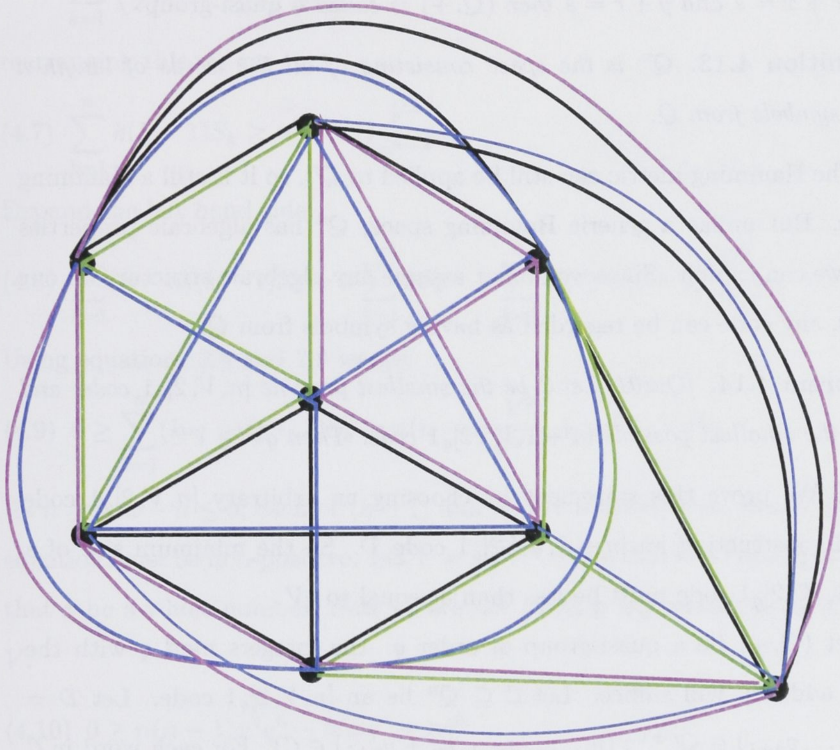

Figure 4.3: Every double edge is in 4 generalized multi-coloured triangles.

4.2 Bounds

We can also use algebraic methods to derive bounds. The following result leads to an upper bound. The proof requires some basic algebra.

Definition 4.12. *Let $(\mathcal{Q}, +)$ be a set of order q with a closed binary operation defined on it. If for every $r, s \in \mathcal{Q}$ there exists unique $x, y \in \mathcal{Q}$ such that $r + x = s$ and $y + r = s$ then $(\mathcal{Q}, +)$ is called a* quasi-group.

Definition 4.13. *$\mathcal{Q}^n$ is the space consisting of all the words of length n with symbols from $\mathcal{Q}$.*

The Hamming metric can still be applied to $\mathcal{Q}^n$, so it is still a Hamming space. But unlike a generic Hamming space, $\mathcal{Q}^n$ has algebraic properties that we can exploit. Since we do not assume any algebraic structure on our codes, any code can be regarded as having symbols from $\mathcal{Q}$.

Theorem 4.14. *[Qui01] Let $\mathcal{C}$ be the smallest possible $[n, V, 2]_q 1$ code, and $\mathcal{D}$ be the smallest possible $[n+1, V', 2]_q 1$ code. Then $qV \geq V'$.*

Proof. We prove this statement by choosing an arbitrary $[n, V, 2]_q 1$ code $\mathcal{C}$ and constructing an $[n+1, qV, 2]_q 1$ code $\mathcal{D}$. So the minimum size of a $[n+1, V', 2]_q 1$ code must be less than or equal to qV.

Let $(\mathcal{Q}, +)$ be a quasi-group of order q: the integers mod q with the usual addition will suffice. Let $\mathcal{C} \subseteq \mathcal{Q}^n$ be an $[n, V, 2]_q 1$ code. Let $\mathcal{D} = \{(v_1, \ldots, v_{n+1}) \in \mathcal{Q}^{n+1} : (v_1, \ldots, v_{n-1}, v_n + v_{n+1}) \in \mathcal{C}\}$. For each word in $\mathcal{C}$ we get q words in $\mathcal{D}$, and so $\mathcal{D}$ is an $[n+1, qV, 2]_q 1$ code.

$\heartsuit$

Given that we have found that the minimum size of a $[3, V, 2]_4 1$ code is 8, we know that the minimum size of a $[4, V, 2]_4 1$ code can be no more than 32.

Corollary 4.15. *[OQW05]*

(4.4) $K_q(n+1, 1, 2) \leq qK_q(n, 1, 2)$

This gives an upper bound. We must also find a lower bound.

Theorem 4.16. *[BEHvL01] If $\mathcal{C}$ is an $[n, V, 2]_q 1$ code then*

(4.5) $V \geq \dfrac{q^{n-1}}{n-1}$

Proof. From Lemma 2.21 we have

(4.6) $\sum_{k=1}^{n} \binom{k}{2} S_k \geq \binom{n}{2} \dfrac{V^2}{q^{n-2}}$

rearranging this we get

(4.7) $\sum_{k=1}^{n} k(k-1)S_k \geq n(n-1)\dfrac{V^2}{q^{n-2}}.$

Expand the left hand side.

(4.8) $\sum_{k=1}^{n}(k-n)(k-1)S_k + n\sum_{k=1}^{n} kS_k - n\sum_{k=1}^{n} S_k \geq n(n-1)\dfrac{V^2}{q^{n-2}}$

Using equations 2.8 and 2.9 we get

(4.9) $0 \geq \sum_{k=1}^{n}(k-n)(k-1)S_k \geq n(n-1)\dfrac{V^2}{q^{n-2}} + nq^n - Vn^2q.$

As $n \geq k$, $k - n \leq 0$, each of $(k-1)$ and S_k are non-negative, so the whole equation must be non-positive. Let $V = xq^{n-1}$. Note that it is not important that x be a whole number, thus we are not making any assumptions about V.

(4.10) $0 \geq n(n-1)x^2q^n - n^2xq^n + nq^n$

Removing the common factor nq^n we get

(4.11) $0 \geq (n-1)x^2 - nx + 1 = (x-1)((n-1)x-1).$

We know from Corollary 2.15 that $V \leq q^{n-1}$, thus $x \leq 1$ and hence $(x-1) \leq 0$. So we need $(n-1)x \geq 1$ so we find that $x \geq \frac{1}{n-1}$ and hence

(4.12) $V \geq \dfrac{q^{n-1}}{n-1}.$

♡

Hence using the notation of our problem

(4.13) $K_q(n, 2, 1) \geq \dfrac{q^{n-1}}{n-1}$

Thus $K_4(4, 2, 1) \geq 21\frac{1}{3}$. $K_4(3, 2, 1) \geq 8$, which is the result obtained in Theorem 3.14. So the bound in Theorem 4.16 is sharp for $n = 3$.

This bound has been improved upon slightly.

Theorem 4.17. *[Qui01] Thm 1. Let $\mathcal{C}$ be an $[n, V, 2]_q 1$ code with $2 \geq n - 1 \geq q \geq 2(n-1)$ then*

(4.14) $V \geq \left\lceil \dfrac{2(q-1)q - 2(n-1) + q}{2(q-1)(n-1) - 2(n-1) + q} q^{n-2} \right\rceil$

This is slightly larger in some cases than Theorem 4.16, though not enough to make a difference in the case we study most, $q = 4$. The proof is long and complex, and so it has been omitted. In the case $q = 2(n-1)$ such as $q = 4$ and $n = 3$, Theorem 4.17 gives the same result as Theorem 4.16.

4.3 More Ideas

We can develop more ideas by splitting the code into 3 parts.

Let $\mathcal{D}_{is} = \{v \in \mathcal{C} \setminus \mathcal{C}_{is} | d(v, \mathcal{C}_{is}) > 2\}$, $\mathcal{E}_{is} = \{v \in \mathcal{C} \setminus \mathcal{C}_{is} | d(v, \mathcal{C}_{is}) = 2\}$

To think of this graphically, $\mathcal{E}_{is}$ contains those vertices which share an $(n-2)$-edge with a member of $\mathcal{C}_{is}$, and $\mathcal{D}_{is}$ contains those vertices which do not.

(4.15) $\mathcal{D}_{is} \cup \mathcal{E}_{is} \cup \mathcal{C}_{is} = \mathcal{C}$

This is a disjoint union.

Lemma 4.18. *[OQW05]Let $\mathcal{C}$ be an $[n, V, d]_q$ code, then*

(4.16) $|\mathcal{D}_{is}| \geq q^{n-1} - p_q(n-1, |\mathcal{C}_{is}|, 2, 1)$

where $p_q(n-1, |\mathcal{C}_{is}|, 2, 1)$ is the maximum size of the ball of radius 1 around an $[n-1, |\mathcal{C}_{is}|, 2]_q$ code.

Proof. For $x \in \mathcal{C} \setminus \mathcal{C}_{is}$, $|B_1(x) \cap H_{is}| = 1$.

For $x \in \mathcal{E}_{is}$, $B_1(x) \cap H_{is} \subset B_1(\mathcal{C}_{is}) \cap H_{is}$ and so

(4.17) $(B_1(\mathcal{C}_{is}) \cap H_{is}) \cup (B_1(\mathcal{D}_{is}) \cap H_{is}) = H_{is}$.

If $u, v \in \mathcal{D}_{is}$ such that $d(u,v) = 1$ then $B_1(u) \cap H_{is} \equiv B_1(v) \cap H_{is}$, and so

(4.18) $|B_1(\mathcal{D}_{is}) \cap H_{is}| \leq |\mathcal{D}_{is}|$.

Putting equations 4.17 and 4.18 together we get

(4.19) $|\mathcal{D}_{is}| \geq |H_{is}| - |B_1(\mathcal{C}_{is}) \cap H_{is}|$

Then use $|H_{is}| = q^{n-1}$ and

(4.20) $|B_1(\mathcal{C}_{is}) \cap H_{is}| \leq p_q(n-1, |C_{is}|, 2, 1)$.

♡

Lemma 4.19. *[OQW05] Let $\mathcal{C}$ be an $[n, V, d]_q R$ code and let i, s be given, then for any j, t*

(4.21) $|(H_{is} \cap H_{jt}) \setminus B_1(\mathcal{C}_{is})| + |\mathcal{C}_{is} \cap \mathcal{C}_{jt}| \leq |\mathcal{C}_{jt}|$.

Proof.

(4.22) $|(H_{is} \cap H_{jt}) \setminus B_1(\mathcal{C}_{is})| + |\mathcal{C}_{is} \cap \mathcal{C}_{jt}| \leq |\mathcal{C}_{jt} \setminus \mathcal{C}_{ir}| + |\mathcal{C}_{jt} \cap \mathcal{C}_{ir}| = |\mathcal{C}_{jt}|$

♡

If we think of $\mathcal{C}_{is}$ as a code $\mathcal{D}$ and $K_{jt_{is}} = K_{is}(\mathcal{D})$ then equation 4.21 is saying

(4.23) $K_{is}(\mathcal{D}) \leq |\mathcal{C}_{jt}|$.

And so the maximum size of $\mathcal{C}_{is}$ for any i, s puts a bound on the size of K_{jt} for any code $\mathcal{C}_{ku}$ of $\mathcal{C}$.

This brings into significance Lemma 3.23 at the end of chapter 3.

Chapter 5

$[4, V, 2]_q$ codes

In this chapter we apply the techniques and results of the previous chapters to the problem of finding $K_4(4, 1, 2)$.

While this problem has already been solved through exhaustive enumeration, it would be useful to find some techniques which do not require such extensive computation. These techniques could then be applied (perhaps with computation) to find results about larger classes of codes.

The author does not prove any new results, but the techniques used are novel. The first proof of Theorem 5.11 [OQW05] uses specific results about codes of smaller sizes. The second original proof uses techniques which should be applicable to codes of various sizes, though this further application has not yet been done.

5.1 Lower Bound

Let $\mathcal{C}$ be a $[4, V, 2]_q$ code with an OQW-associated graph. From Corollary 4.11 we know that

$$(5.1)\quad |B_1(\mathcal{C})| \leq (1 + 4(q-1))V - \frac{2}{3}E_2.$$

Thus finding ways of counting and bounds on the number of double edges and generalized multi-coloured triangles will be useful.

Lemma 5.1. *Let $\mathcal{C}$ be a $[4,V,2]_4$ code with an OQW-associated graph. Let $|\mathcal{C}_{is}| = zq + w$ where $z \geq 0$ and $0 \leq w < q$. Then $\mathcal{H}_{is}$ will contain at least*

$$\frac{q}{2}(z-1)z + wz \tag{5.2}$$

double edges with colours i and j.

Proof. Each $\mathcal{H}_{is}$ is the graph associated with a $[3,|\mathcal{C}_{is}|,2]_4$ (in colours j,k,l), code coupled with a complete graph on $|\mathcal{C}_{is}|$ vertices of colour i. From Corollary 2.10 we know that there must be at least $\frac{q}{2}(z-1)z + wz$ edges of colour j in $\mathcal{H}_{is}$ and those edges must be coupled with an edge of colour i.

♡

By looking at the possible values of $\mathcal{C}_{is}$ we can determine the minimum number of double edges for a graph OQW-associated with a $[4,V,2]_4$ code. Thus we now need to look at some bounds to determine what values of V and $\mathcal{C}_{is}$ are worth further investigation.

From Theorem 4.16 we know that $K_4(4,1,2) \geq 22$.

Remembering that $K_q(n,1,2) \geq K_q(n,1)$ (equation 1.1) we first look at $K_4(4,1)$ to provide a better lower bound.

The following is a slightly weaker version of Lemma 4.18,

Theorem 5.2. *[SK69a] Theorem 2. There exists $\alpha \leq \frac{K_q(4,1)}{q}$ such that*

$$K_q(4,1) \geq q^3 + \alpha - p_q(3,\alpha,d,1). \tag{5.3}$$

Proof. Follow proof of Lemma 4.18, letting $\alpha \leq |\mathcal{C}_{is}|$ and $|\mathcal{D}_{is}| \geq q^3 - p_q(3,|\mathcal{C}_{is}|,d,1)$.

♡

Theorem 5.3. *[SK69a] Theorem 3 and [SHK69] Theorem 3.*

$$K_4(4,1) = 24 \tag{5.4}$$

Proof. Let $\alpha = 5$ then from Theorem 5.2 and Table 3.1

(5.5) $K_4(4,1) \geq 4^3 + 5 - 45 = 24.$

If $\alpha = 6$, then equation 5.3 will be false. If $\alpha < 5$, then the bound will be lower.

This can be achieved by the code

0000	1001	2023	3032
0011	1010	2123	3132
0101	1100	2230	3203
0110	1111	2231	3213
0222	1222	2302	3321
0333	1333	2312	3320

♡

It is now known through exhaustive search [OQW05] that this is the only $[4, 24]_4 1$ code, however its minimum distance is certainly not 2, and hence it is not a $[4, 24, 2]_4 1$ code as we are looking for.

However from this we do now know that

(5.6) $K_4(4,1,2) \geq 24$

which is a better lower bound than given by Theorem 4.16.

5.2 Properties of $[4, V, 2]_4 1$ Codes

Lemma 5.4. *[OQW05] Let $\mathcal{C}$ be a $[4, V, 2]_4 1$ code then*

1. *if there exist i, s such that $|\mathcal{C}_{is}| \leq 4$ then $V \geq 28$.*

2. *if there exist i, s such that $|\mathcal{C}_{is}| = 5$ then $V \geq 25$.*

Proof. 1. From table 3.1 we know that $p_4(3,4,1) = 40$. From Lemma 4.18, if $|\mathcal{C}_{is}| \leq 4$ then,

(5.7) $|\mathcal{D}_{is}| \geq 64 - 40 = 24.$

Since

$$(5.8)\quad V = |\mathcal{D}_{is}| + |\mathcal{E}_{is}| + |\mathcal{C}_{is}| = 24 + |\mathcal{E}_{is}| + 4 \geq 28.$$

2. Let $|\mathcal{C}_{is}| = 5$. From table 3.1 we know that $p_4(3, 5, 2, 1) = 45$, and hence $|B_1(\mathcal{C}_{is}) \cap H_{is}| \leq 45$. Using Lemma 4.18, if $|B_1(\mathcal{C}_{is}) \cap H_{is}| < 44$ then $|D_{ir}| \geq 64 - 44 = 20$, and hence $|\mathcal{C}| \geq 20 + 5$.

If $|B_1(\mathcal{C}_{is}) \cap H_{is}| = 45$ then using table 3.1 and without loss of generality

$$(5.9)\quad \mathcal{C}_{10} = \{0000, 0011, 0101, 0222, 0333\}.$$

This yields

$$(5.10)\quad (H_{10} \cap H_{20}) \setminus B_1(\mathcal{C}_{10}) = \{0023, 0032\}.$$

If $\mathcal{E}_{10} \neq \emptyset$ then

$$(5.11)\quad V = |\mathcal{D}_{10}| + |\mathcal{E}_{10}| + |\mathcal{C}_{10}| \geq 19 + 1 + 5 = 25,$$

or if $\mathcal{D}_{10} \geq 20$ then $V \geq 20 + 5 = 25$.

Otherwise assume $E_{10} = \emptyset$

$$(5.12)\quad |\mathcal{C}_{20}| = |\mathcal{C}_{10} \cap \mathcal{C}_{20}| + |\mathcal{D}_{10} \cap \mathcal{C}_{20}|$$

Words in $v \in \mathcal{D}_{10} \cap \mathcal{C}_{20}$ must have a distance of at least 3 from any word in $\mathcal{C}_{10}$. They will be of structure $v = r_1 0 r_2 r_3$. Thus $r_1 \in \{1, 2, 3\}$, and $r_2, r_3 \in \{2, 3\}$. Since $d(v, 0222) \geq 3$ and $d(v, 0333) \geq 3$ we find that $r_2 \neq r_3$. Hence there are at most 2 words in $\mathcal{D}_{10} \cap \mathcal{C}_{20}$; $r_1 023$ and $r_1' 032$ where $r_1, r_1' \in \{1, 2, 3\}$. We already know that $|\mathcal{C}_{10} \cap \mathcal{C}_{20}| = 2$ and so from equation 5.12

$$(5.13)\quad |\mathcal{C}_{20}| = |\mathcal{C}_{10} \cap \mathcal{C}_{20}| + |\mathcal{D}_{10} \cap \mathcal{C}_{20}| \leq 2 + 2 = 4$$

from part 1, we know that if $|\mathcal{C}_{is}| \leq 4$ then $V \geq 28$.

♡

Corollary 5.5. *[OQW05] If $\mathcal{C}$ is a $[4, 24, 2]_4 1$ code then $|\mathcal{C}_{is}| = 6$ for all i, s.*

Corollary 5.6. *If $\mathcal{C}$ is a $[4, V, 2]_4 1$ code then $V \geq 24$.*

Although we already know this from Theorem 5.3, the above statements provide another proof.

5.3 Properties of $[4, 24, 2]_4 1$ Codes

Lemma 5.7. *If $\mathcal{G}$ is a graph OQW-associated to $\mathcal{C}$ a $[4, 24, 2]_4 1$ code, then $\mathcal{G}$ must contain at least 48 double edges.*

Proof. From Corollary 5.5 we know that $|\mathcal{G}_{is}| = 6$ for all i, s. Each $\mathcal{G}_{is}$ is a K_6 of colour i coupled with a graph associated with $\mathcal{D}$ a $[3, 6, 2]_4$ code in colours j, k, l. From Corollary 2.9 the graph of $\mathcal{D}$ must have 2 edges of each colour. Thus each $\mathcal{G}_{is}$ must have 2 double edges with colours i, j. Given that every edge of colour i is in $\mathcal{G}_i$, $\mathcal{G}$ must have at least $4 \times 2 = 8$ double edges of colours i, j. There are $\binom{4}{2} = 6$ possible combinations of edge colours. Thus there must be at least $8 \times 6 = 48$ double edges in $\mathcal{G}$. ♡

Lemma 5.8. *Let $\mathcal{G}$ be a graph OQW-associated with a $[4, 24, 2]_4 1$ code with 48 double edges. If $\mathcal{H}_{is}$ contains 4 generalized multi-coloured triangles then there are 2 vertices u, v with double degree 0 in $\mathcal{H}_{is}$ and any $\mathcal{H}_{js}$ that are adjacent to u and v may contain a maximum of 2 generalized multi-coloured triangles.*

Proof. Without loss of generality we work with $\mathcal{H}_{10}$ and $v = 0000$.

If $\mathcal{H}_{10}$ contains 4 generalised multi-coloured triangles, then without loss of generality $\mathcal{C}_{10} = \{0000, 0111, 0222, 0233, 0323, 0332\}$.

If we are to cover all words in $H_{10} \cap H_{20}$ (*i.e.* words of the form 00**), and 0000 has no double edges in $\mathcal{H}_{10}$, then all words of the form $00**$ must be covered by a word in $\mathcal{C}_{10}$ or $\mathcal{C}_{20}$.

The words in $H_{10} \cap H_{20}$ not covered by $\mathcal{C}_{10}$ are $\{0012, 0021, 0013, 0031\}$. Thus $\mathcal{C}_{20} = \{0000, *012, *021, *013, 0031, *0**\}$, where $* \in \{0, 1, 2, 3\}$.

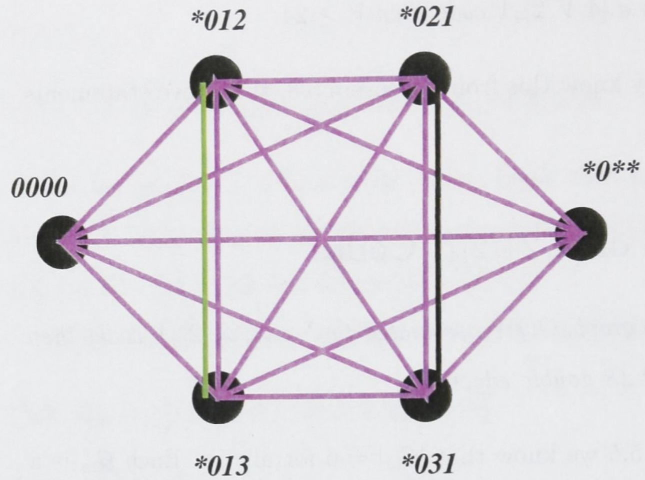

Figure 5.1: $\mathcal{H}_{20}$

If we look at the graph of $\mathcal{C}_{20}$, drawing in only the edges that are defined so far we get figure 5.1. If the green edge (colour 3) is to be in 2 generalized multi-coloured triangles then both vertices $*012$ and $*013$ must be incident with a black edge (colour 4). This would require either a triple edge (green, black and purple), contradicting $d = 2$; or 3 black edges contradicting $\mathcal{G}$ has 48 double edges. The same applies to the black edge, and thus $\mathcal{H}_{20}$ may have a maximum of 2 generalised multi-coloured triangles.

This can be done for $\mathcal{C}_{30}$, $\mathcal{C}_{40}$, with the same result. Then taking $u = 0111$, the process can be followed for $\mathcal{C}_{21}$, $\mathcal{C}_{31}$ and $\mathcal{C}_{41}$, completing the proof. ♡

Lemma 5.9. *Let $\mathcal{G}$ be a graph OQW-associated to $\mathcal{C}$ a $[4, 24, 2]_4 1$ code such that $\mathcal{G}$ has 48 double edges. Then $\mathcal{H}_{is}$ can be adjacent to at most 1 $\mathcal{H}_{jt}$ such that $\mathcal{H}_{jt}$ contains 4 generalized multi-coloured triangles and $|\mathcal{H}_{jt} \cap \mathcal{H}_{is}| = 1$.*

Proof. Let $\mathcal{H}_{10}$ have 4 generalized multi-coloured triangles. Then without loss of generality $\mathcal{C}_{10} = \{0000, 0011, 0101, 0110, 0333, 0222\}$. And from Lemma 5.8 $\mathcal{C}_{22} = \{0222, *213, *231, *230, *203, *2**\}$.

Assume $\mathcal{H}_{31}$ contains 4 generalized multi-coloured triangles then $|B_1(\mathcal{C}_{31}) \cap$

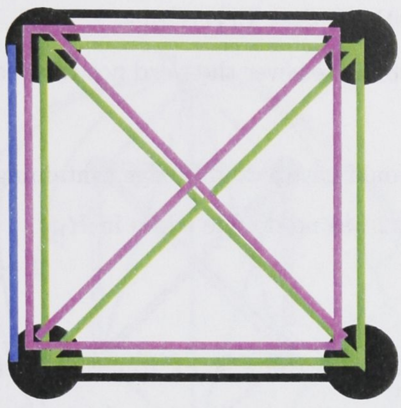

Figure 5.2: $\mathcal{C}_{31}$ must cover 3 of these words

$H_{32} \cap H_{22}| = 12$. Thus the words in $H_{22} \cap H_{31}$ not covered by $\mathcal{C}_{31}$ must be covered by $\mathcal{C}_{22}$.

$*213 \in \mathcal{C}_{22} \cap \mathcal{C}_{31}$ and so will cover whatever word is covered by $*203$. Thus the 4 words in $H_{22} \cap H_{31}$ not covered by $\mathcal{C}_{31}$ must be $\{0212, x211, y210, z2{*}{*}\}$. Given that $0222 \in \mathcal{C}_{10} \cap \mathcal{C}_{22}$ cannot have any double edges with colour 1, $x, y, z \neq 0$ and to avoid a triple edge $x \neq y$. Thus without loss of generality the words not covered by $\mathcal{C}_{31}$ are$\{0212, 2210, 3211, *2**\}$. Which means that $\mathcal{C}_{31}$ must cover exactly 3 of $\{0210, 3210, 0211, 2211\}$, and that $\mathcal{C}_{31}$ must contain exactly 3 of $\{0*10, 3*10, 0*11, 2*11\}$,

These 4 words have a graph like figure 5.2.

Any way of choosing 3 words will result in 2 of the words sharing a double edge in $\mathcal{H}_{31}$, and hence they must be in generalized multi-coloured triangles. Given the structure of $\mathcal{H}_{31}$ has 4 generalized multi-coloured triangles, we would require either $1*11$, $2*11$ or $3*11 \in \mathcal{C}_{31}$ (or $1*10$, $2*21$ or $3*10$ if $0*10 \in \mathcal{C}_{31}$. If $2*11$, then 2211 is covered, and hence $2*11$ must have been one of the chosen 3 words; if $3*11$, then 3211 is covered- which by previous paragraph it should not be; if $1 * 11$, then 1213 has a double edge with colour 1 which contradicts the hypothesis that $|\mathcal{C}_{13} \cap \mathcal{C}_{22}| = 1$.

Thus $\mathcal{C}_{22}$ must contain one of $02{*}0$ or $02{*}1$ to cover the word not covered by $\mathcal{C}_{31}$.

We know that $0222 \in \mathcal{C}_{22}$, thus 0222 would have a double edge containing colour 1 (blue). This contradicts that 0222 has no double edges in $\mathcal{H}_{10}$.

♡

Lemma 5.10. *Let $\mathcal{G}$ be a graph OQW-associated to a $[4, 24, 2]_4 1$ code. Let $\mathcal{G}$ have 48 double edges. If there are 16 generalized multi-coloured triangles with base colour i, then there are no multi-coloured triangles with any other base colour.*

Proof. Without loss of generality let there be 16 generalized multi-coloured triangles of base colour 1 (blue). We begin by drawing $\mathcal{H}_{1s}$ for each value of s. We know from Lemma 3.12 that the only way for this is to have multi-coloured K_4 coupled with the blue K_6 for each $\mathcal{H}_{1s}$.

There are 2 possibilities for a multi-coloured triangle between the $\mathcal{H}_{1s}$. 2 vertices must be from the same $\mathcal{H}_{is}$, because we need a double edge involving colour 1 (blue) *i.e.* vertices a and b . Then the third must be from $\mathcal{H}_{1t}$, and can either be incident with 3 double edges in $\mathcal{H}_{1t}$ *i.e.* vertex c; or incident with 0 double edges in $\mathcal{H}_{it}$ *i.e.* vertex d.

Let us try to form a generalised multi-coloured triangle on vertices a, b, c, figure 5.3. To do this we must add in sufficient edges to form purple, green and black 6-cliques. The purple, and green have been filled in. The only vertices shared by the purple and green 6-cliques are a and b, otherwise more double edges would be formed, contradicting that $\mathcal{G}$ has 48 double edges.

To form the black 6-clique, 2 of vertices u, v, w or x must be used. The dotted and dashed lines show that using any of these will lead to a double

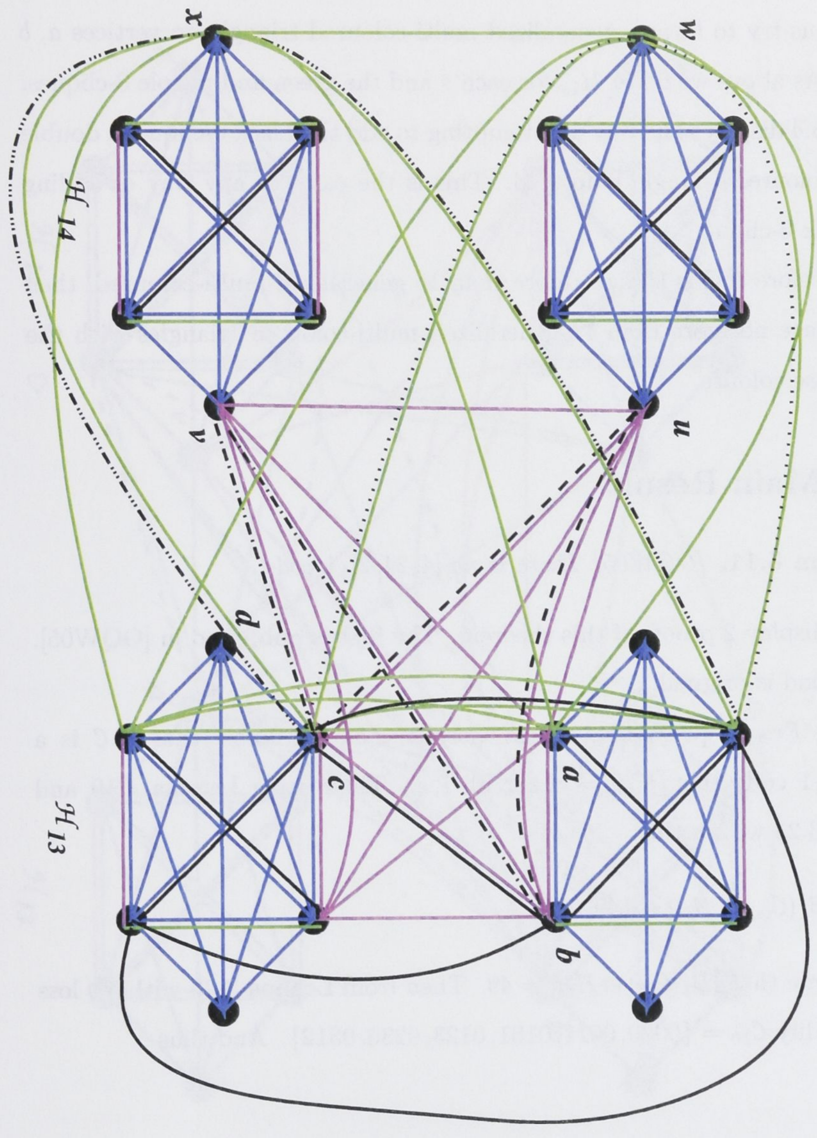

Figure 5.3: There are 16 generalized multi-coloured triangles with a blue base.

single coloured triangle. This contradicts that we have the minimum number of edges.

Let us try to form a generalized multi-coloured triangle on vertices a, b and d. As above we fill in $\mathcal{H}_{1s}$ for each s and the green and purple 6-cliques. Figure 5.4 displays one way of attempting to add the black 6-clique, a double single coloured triangle is formed. This is the case for any way of adding the black 6-clique.

Therefore if $\mathcal{G}$ is to have more than 16 generalized multi-coloured, then it can have no more than 14 generalized multi-coloured triangles with the same base colour. $\heartsuit$

5.4 Main Result

Theorem 5.11. *[OQW05] There is no* $[4, 24, 2]_4 1$ *code.*

We display 2 proofs of this theorem. The first is published in [OQW05]. The second is original.

First Proof. [OQW05]. From Corollary 5.5 we know that if $\mathcal{C}$ is a $[4, 24, 2]_4 1$ code then $|\mathcal{C}_{is}| = 6$ for all i, s. Then using Lemma 4.19 and Lemma 3.23 we see that

$$(5.14)\quad |B_1(\mathcal{C}_{is}) \cap H_{is}| \in \{49, 52\}$$

Assume that $|B_1(\mathcal{C}_{is}) \cap H_{is}| = 49$. Then from Lemma 3.23 without loss of generality $\mathcal{C}_{10} = \{0000, 0011, 0101, 0123, 0230, 0312\}$. And thus

$$(5.15)\quad (H_{10} \cap H_{23}) \setminus B_1(\mathcal{C}_{10}) = \{0303, 0320, 0321, 0331, 0333\}.$$

Now we know that $|\mathcal{C}_{23}| = 6$, and because $d = 2$, $|B_1(\mathcal{C}_{10}) \cap B_1(\mathcal{C}_{23})| \leq 1$. So there exist $r_1, r_2, r_3, r_4, r_5 \in \{1, 2, 3\}$ such that

$$(5.16)\quad \mathcal{C}_{23} = \{0312, r_1 303, r_2 320, r_3 321, r_4 331, r_5 333\}$$

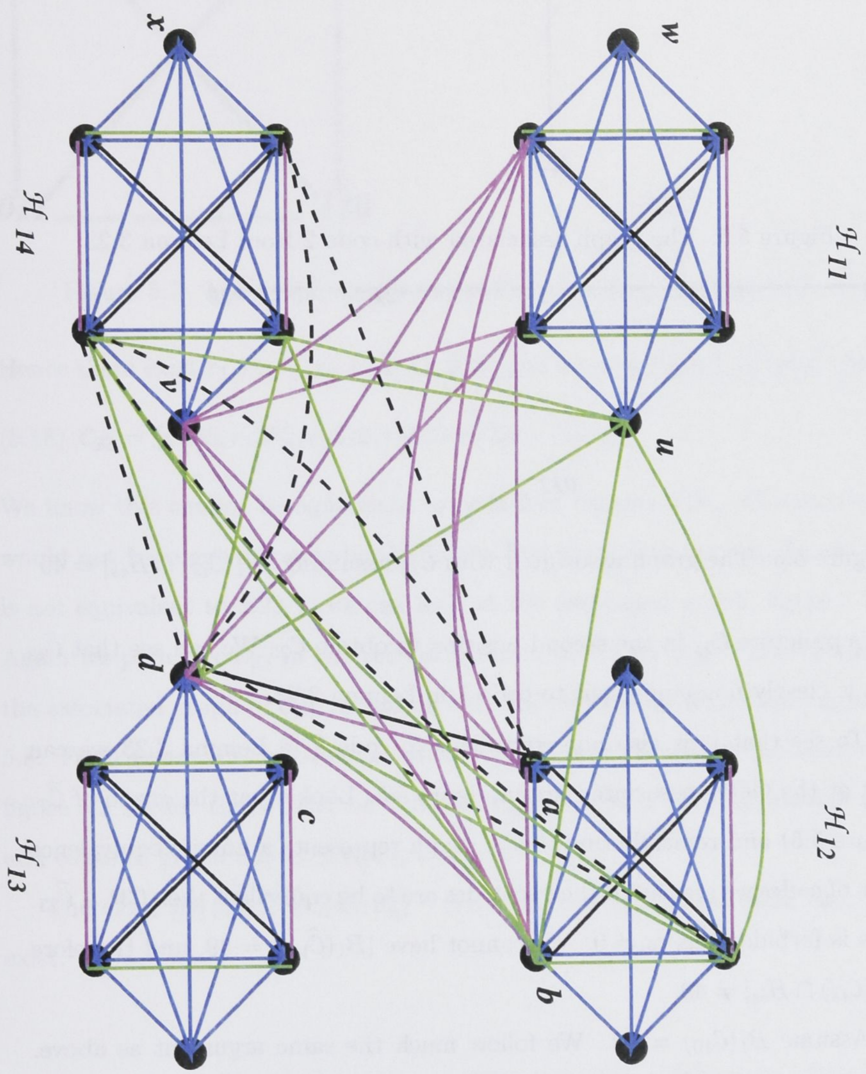

Figure 5.4: There are 16 generalized multi-coloured triangles with a blue base.

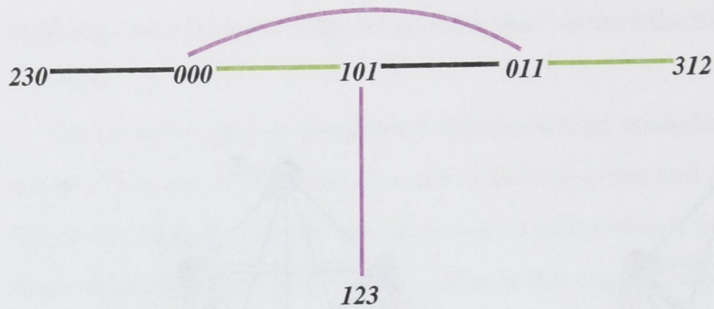

Figure 5.5: The graph associated with code 2 from Lemma 3.23

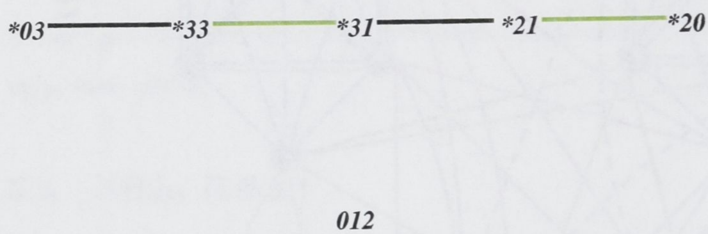

Figure 5.6: The graph associated with $\widehat{\mathcal{C}}_{23}$ assuming $|B_1(\mathcal{C}_{is}) \cap H_{is}| = 49$

Let's puncture $\mathcal{C}_{23}$ in the second position to obtain $\widehat{\mathcal{C}}_{23}$ We can see that $\widehat{\mathcal{C}}_{23}$ this is clearly non-equivalent to code 1 in Lemma 3.23.

To see that it is also non-equivalent to code 2 in Lemma 3.23 we can look at the OQW-associated graph, figure 5.5. Looking at the graph of $\widehat{\mathcal{C}}_{23}$ (figure 5.6) and remembering that a graph represents a unique equivalence class of codes we see that if the two codes are to be equivalent then $031 \in \widehat{\mathcal{C}}_{23}$. This is forbidden as $r_4 \neq 0$. We cannot have $|B_1(\widehat{\mathcal{C}}_{10})| = 49$, and therefore $|B_1(\mathcal{C}_{is}) \cap H_{is}| \neq 49$.

Assume $B_1(\mathcal{C}_{10}) = 52$. We follow much the same argument as above. Then from Lemma 3.23 without loss of generality

$$\mathcal{C}_{10} = \{0000, 0011, 0101, 0110, 0222, 0333\}.$$

And thus

$$(5.17)\quad (H_{10} \cap H_{23}) \setminus B_1(\mathcal{C}) = \{0302, 0312, 0320, 0321\}.$$

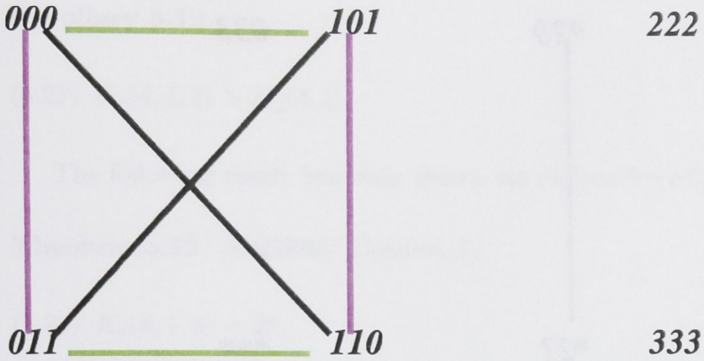

Figure 5.7: The graph associated with code 1 from Lemma 3.23

Hence there exist $r_1, r_2, r_3, r_4, r_5 \in \{1, 2, 3\}$ and $s_1, s_2 \in \{0, 1, 2, 3\}$ such that

$$(5.18)\quad \mathcal{C}_{23} = \{0333, r_1 302, r_2 312, r_3 320, r_4 321, r_5 3 s_1 s_2\}$$

We know this cannot be equivalent to code 2 of Lemma 3.23, otherwise we would not have got a contradiction in the first part of the proof. To see it is not equivalent to 3.23 1, we can look at the associated graph, figure 5.7. Again we puncture $\mathcal{C}_{23}$ in the second position to obtain $\widehat{\mathcal{C}_{23}}$. Constructing the associated graph of $\widehat{\mathcal{C}_{23}}$, including only the edges that are certain, figure 5.8. From figure 5.7, we need 2 vertices which have no edges. Looking at figure 5.8 we see these must be 033 and r_5, s_1, s_2. It is then impossible to add edges to graph 5.8 to create graph 5.7.

Therefore $|B_1(\mathcal{C}_{23})| \notin \{49, 52\}$. And hence a $[4, 24, 2]_4 1$ code cannot exist.

♡

Second Proof. The Hamming space $H(4,4)$ has $4^4 = 256$ words. Thus if $\mathcal{C}$ has a covering radius of 1, then $|B_1(\mathcal{C})| = 256$. We show that

$$(5.19)\quad |B_1(\mathcal{C})| \leq 254.$$

From Lemma 5.8 we know that if $\mathcal{H}_{is}$ contains 4 generalised multi-coloured triangles, then there are 6 $\mathcal{H}_{jt}$ that contain at most 2 generalized

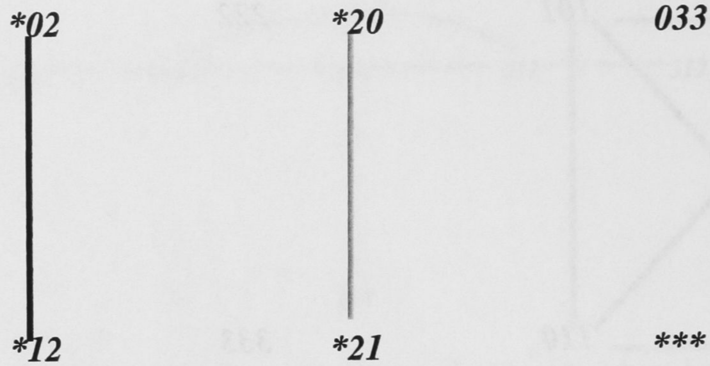

Figure 5.8: The graph associated with $\widehat{\mathcal{C}}_{23}$ assuming $|B_1(\mathcal{C}_{is}) \cap H_{is}| = 52$

multi-coloured triangles.

From Lemma 5.9 we know that if there is $\mathcal{H}_{is}$ and $\mathcal{H}_{jt}$ such that $j \neq i$ and both have 4 generalized multi-coloured triangles, then the 6 $\mathcal{H}_{ku}$'s specified in Lemma 5.8 cannot be the same. And so these can be the only $\mathcal{H}_{is}$ with 4 generalized multi-coloured triangles.

From Lemma 5.10 we know that if there are more than 16 generalized multi-coloured triangles in $\mathcal{G}$, then there can be at most 3 $\mathcal{H}_{is}$ of the same colour that contain 4 generalized multi-coloured triangles.

Hence there can be at most 3 $\mathcal{H}_{is}$ of any colour base that contain 4 generalized multi-coloured triangles. The rest may contain a maximum of 2. Thus there can be a maximum of 38 generalized multi-coloured triangles in $\mathcal{G}$. From Theorem 4.8,

(5.20) $|B_1(\mathcal{C})| \leq 13 \times 24 - 2 \times 48 + 38 = 254.$

We know from Corollary 4.11 that adding extra edges to $\mathcal{G}$ can only decrease the possible size of $B_1(\mathcal{C})$, and hence the maximum size of a ball of radius 1 around a $[4, 24, 2]_4$ code is 254.

$|H(4,4)| = 256$, and so no $[4, 24, 2]_4$ code can cover $H(4,4)$.

♡

Corollary 5.12.

(5.21) $K_q(4,1,2) > K_q(4,1)$

The following result has been shown via exhaustive computer search.

Theorem 5.13. *[OQW05] Theorem 11.*

(5.22) $K_4(4,1,2) = 28$

and there are exactly 2 non-equivalent $[4,28,2]_4 1$ *codes.*

Perhaps further investigation of graphs and association methods may yield this result without the need for an exhaustive search.

Chapter 6

Conclusion

The original aim for this work was to find a proof of that $K_4(4, 1, 2) = 28$ (Theorem 5.13) that does not require exhaustive search; and in doing so develop techniques which may be used to solve similar problems. Whilst the entire aim was not achieved, a new proof that $K_4(4, 1, 2) > K_4(4, 1)$ has been found and interesting techniques and results have been developed.

6.1 Techniques

The OQW-association method defined by Ostergaard, Quistorff and Wassermann [OQW05] has been investigated. Theorem 2.6 gives a set of necessary and sufficient conditions for a simple graph to be OQW-associated with a code and Theorem 2.14 gives the conditions for a multi-graph. Investigating the properties of graphs which meet the conditions leads results about the OQW-associated codes (Theorem 3.9).

Techniques for counting multi-coloured triangles within a simple graph are developed. The concept of a generalised multi-coloured triangle is defined for multi-graphs and counting techniques are developed.

These techniques are used to provide bounds on the maximum size of $B_1(\mathcal{C})$ for $\mathcal{C}$ an $[n, V, d]_q$ code (Theorem 4.11), with sharper bounds for

$[n, V, n-1]_q$ codes (equation 3.7).

The BEHL-association method of Blokhuis, Enger, Hollmann and van Lint ([BEHvL01]) has been presented. No further investigation of this association method has been done.

6.2 Results

Results on $[3, V, 2]_4$ codes are summarised in Table 3.1. Of particular note is Theorem 3.16, which shows there is no $[3, 9, 2]_4 1$ code, contrasting with Theorems 3.14 and 3.18 showing that both $[3, 8, 2]_4 1$ and $[3, 10, 2]_4 1$ codes exist.

A new proof is given that no $[4, 24, 2]_4 1$ code exists (Theorem 5.11) and therefore $K_4(4, 1) < K_4(4, 1, 2)$.

6.3 Future Applications

Chapter 3 concludes with the open question of determining if $[3, V, 2]_4 1$ codes exist for $V \in \{11, 12, 13, 14, 15\}$. This is the most obvious direction to continue in. However there may be more fruitful areas. In the case of $[3, V, 2]_4$ of chapter 3 may be applied to $q > 4$. Is $K_5(4, 1) < K_5(4, 1, 2)$?

Further investigation of the properties of graphs used in the OQW association methods may lead to new results about covering codes. Techniques for counting multi-edges may provide a sharper bound than Theorem 4.11. Investigating the properties of graphs which do meet the bound would be interesting both from a graph theoretic point of view as well as the applications to existence of codes.

The methods for counting multi-coloured triangles may be employed with appropriate software to enhance computer aided investigations into covering codes.

Index

Bibliography

[BEHvL01] A. Blokhuis, S. Enger, H.D.L. Hollmann, and J.H. van Lint. On codes with covering radius 1 and minimum distance 2. *Indagationes Mathematical*, 12(4):449–452, 2001.

[Bol79] B. Bollobás. *Graph Theory: An Introductory Course.* Springer-Verlag, 1979.

[CHLL97] G. Cohen, I. Honkala, S. Lytsin, and A. Lobstein. *Covering Codes.* North Holland, 1997.

[Hil86] R. Hill. *A First Course in Coding Theory.* Oxford University Press, 1986.

[HP03] W. Huffman and V. Pless. *Fundamentals of Error-Correcting Codes.* Cambridge University Press, 2003.

[NS63] E. A. Nordhaus and B. M. Stewart. Triangles in an ordinary graph. *Canadian Journal of Mathematics*, 15:33–41, 1963.

[OQW05] Patric R. J. Ostergard, Jorn Quistorff, and Alfred Wassermann. New results on codes with covering radius 1 and minimum distance 2. *Designs, Codes and Cryptography*, 35:241–250, 2005.

[Qui01] J. Quistorff. On codes with a given minimum distance and covering radius. *Beiträge zur Algebra und Geometrie*, 42(2):601–611, 2001.

[SHK69] R.G. Stanton, J.D. Horton, and J.G. Kalbfleisch. Covering theorems for vectors with special reference to the case of four and five components. *Journal of the London Mathematical Society*, pages 493–499, 1969.

[SK69a] R.G. Stanton and J.G. Kalbfleisch. A combinatorial problem in matching. *Journal of the London Mathematical Society*, pages 60–64, 1969.

[SK69b] R.G. Stanton and J.G. Kalbfleisch. Corrigendum: A combinatorial problem in matching. *Journal of the London Mathematical Society*, page 398, 1969.

[Spe06] Ted Spence. Strongly regular graphs on at most 64 vertices. www.maths.gla.ac.uk/∼es/srgraphs.html, 1/12/2006.

[Wil79] R.J. Wilson. *Introduction to Graph Theory*. Longman Group Limited, second edition, 1979.